

SMALL PERMUTATION CLASSES

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We establish a phase transition for permutation classes (downsets of permutations under the permutation containment order): there is an algebraic number κ , approximately 2.20557, for which there are only countably many permutation classes of growth rate (Stanley-Wilf limit) less than κ but uncountably many permutation classes of growth rate κ , answering a question of Klazar. We go on to completely characterize the possible sub- κ growth rates of permutation classes, answering a question of Kaiser and Klazar. Central to our proofs are the concepts of generalized grid classes (introduced herein), partial well-order, and atomicity (also known as the joint embedding property).

1. INTRODUCTION

For any collection (also known as a *property*), \mathcal{P} , of finite combinatorial structures, the function which maps n to the number of structures in \mathcal{P} with ground set $[n] = \{1, 2, \dots, n\}$ is known as the *speed* of \mathcal{P} . A property \mathcal{P} is further said to be *hereditary* if it is closed under taking sub-structures. The study of speeds of hereditary properties of combinatorial structures dates back to Scheinerman and Zito [26], who studied labeled graphs.

Our interest lies with hereditary properties of permutations, which we call *permutation classes*. A permutation π of $[n]$ ¹ contains the permutation σ of $[k]$ (written $\sigma \leq \pi$) if π has a subsequence of length k order isomorphic to σ . For example, $\pi = 391867452$ (written in list, or one-line notation) contains $\sigma = 51342$, as can be seen by considering the subsequence $91672 (= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$. A permutation class is then a downset

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¹Here $[n] = \{1, 2, \dots, n\}$ and, more generally, for $a, b \in \mathbb{N}$ ($a < b$), the interval $\{a, a + 1, \dots, b\}$ is denoted by $[a, b]$, the interval $\{a + 1, a + 2, \dots, b\}$ by $(a, b]$, and so on.

of permutations under this order; thus if \mathcal{C} is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. For any set X of permutations, we define its *closure* to be the permutation class $\{\sigma : \sigma \leq \pi \text{ for some } \pi \in X\}$. For any permutation class \mathcal{C} there is a unique (and possibly infinite) antichain B such that $\mathcal{C} = \text{Av}(B) = \{\pi : \pi \not\geq \beta \text{ for all } \beta \in B\}$. This antichain B is called the *basis* of \mathcal{C} .

We denote by \mathcal{C}_n ($n \in \mathbb{N}$) the set of permutations in \mathcal{C} of length n , so the speed of \mathcal{C} is the function $n \mapsto |\mathcal{C}_n|$. We further refer to $\sum |\mathcal{C}_n| x^n$ as the *generating function* for \mathcal{C} . (Whether this generating function counts the empty permutation is a matter a taste; we elect to count it except when noted.)

The Markus-Tardos Theorem [23] (formerly the Stanley-Wilf Conjecture) states that all permutation classes other than the class of all permutations have at most exponential speed, i.e., for every class \mathcal{C} with a nonempty basis, there is a constant K so that \mathcal{C} contains at most K^n permutations of length n for all n . Thus every nondegenerate permutation class \mathcal{C} has finite *upper* and *lower growth rates* defined, respectively, by

$$\begin{aligned} \overline{\text{gr}}(\mathcal{C}) &= \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|} \text{ and} \\ \underline{\text{gr}}(\mathcal{C}) &= \liminf_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}. \end{aligned}$$

It is only conjectured that every permutation class has a *growth rate*, but when we are dealing with a class for which $\overline{\text{gr}}(\mathcal{C}) = \underline{\text{gr}}(\mathcal{C})$, we denote this quantity by $\text{gr}(\mathcal{C})$.

Herein we are concerned with the set of (upper, lower) growth rates of permutation classes. From this viewpoint, the Erdős-Szekeres Theorem below characterizes the permutation classes of growth rates 0: if \mathcal{C} contains arbitrarily long monotone permutations then $\text{gr}(\mathcal{C}) \geq 1$ while otherwise \mathcal{C} is finite and so $\text{gr}(\mathcal{C}) = 0$.

The Erdős-Szekeres Theorem [13]. *Every permutation π of length $(m-1)^2 + 1$ contains a monotone permutation of length at least m .*

Moving beyond the growth rate 0 classes, Kaiser and Klazar [21] proved that the only growth rates of permutation classes (they are in this case proper, unadjectivated, growth rates) less than 2 are positive solutions to $1 - 2x^k + x^{k+1}$ for some $k \geq 0$, i.e., that the speeds of such classes are logarithmically asymptotic to the k -Fibonacci numbers for some k . Later, Klazar [22] showed that there are only countably many permutation classes with such growth rates. Two natural questions are then

1. What is the next (lower, upper, proper) growth rate after 2? (Asked by Kaiser and Klazar [21].)
2. What is the smallest (lower, upper, proper) growth rate for which there are uncountably many permutation classes? (Asked by Klazar [22].)

We provide answers to both of these: Answer 1 is

$$\nu = \text{the unique positive root of } 1 + 2x + x^2 + x^3 - x^4 \approx 2.06599$$

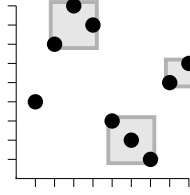


Figure 1: The plot of 479832156, an inflation of 2413.

(this is Theorem 6.4) while Answer 2 is

$$\kappa = \text{the unique positive root of } 1 + 2x^2 - x^3 \approx 2.20557$$

(Theorem 5.10). Furthermore, as established in Theorem 6.4, κ is the least accumulation point of accumulation points of growth rates of permutation classes.

Central to our proofs is the concept of generalized grid classes, introduced in Section 2. In Section 3 we present a characterization of these classes and of grid irreducible classes. We then show, in Section 4, that small permutation classes lie in particularly tractable grid classes, which allows us in Section 5 to provide Answer 2, i.e., that there are only countably many permutation classes with growth rate less than κ , and as a consequence, that each of these classes is partially well-ordered. This partial well-order condition turns out to be essential for the characterization of growth rates below κ (these all happen to be proper growth rates), which follows in Section 6.

Quite often we need to show that small permutation classes cannot contain certain types of structures, or in other words, that if a class were to contain those structures then it would be large. These computations, which are in the cases we need them mostly routine but always tedious, have been relegated to the Appendix.

The remainder of the introduction consists of basic structural notions used in the proof. For completeness, we have liberally included short proofs of the lesser known results. Also, for concreteness, we have tended to specialize these results to the case of permutations, although in many cases they hold much more generally.

Inflations, simple permutations, and wreath closures. An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous. Every permutation π of $[n]$ has intervals of length 0, 1, and n ; π is said to be *simple* if it has no other intervals. For an extensive study of simple permutations we refer the reader to Brignall's thesis [11] and survey article [10].

To go in the other direction, given $\sigma \in S_m$ and nonempty permutations $\alpha_1, \dots, \alpha_m$, the *inflation* of σ by $\alpha_1, \dots, \alpha_m$ — denoted $\sigma[\alpha_1, \dots, \alpha_m]$ — is the permutation obtained by replacing each entry $\sigma(i)$ by an interval that is order isomorphic to α_i . For example, $2413[1, 132, 321, 12] = 4\ 798\ 321\ 56$ (see Figure 1). Note that simple permutations clearly cannot be deflated, and conversely,

Proposition 1.1 (Albert and Atkinson [1]). *Every permutation except 1 is the inflation of a unique simple permutation of length at least 2.*

A class \mathcal{C} of permutations is *wreath-closed* if $\sigma[\alpha_1, \dots, \alpha_m] \in \mathcal{C}$ for all $\sigma, \alpha_1, \dots, \alpha_m \in \mathcal{C}$. The *wreath-closure* of a set X , denoted $\mathcal{W}(X)$, is defined as the smallest wreath-closed class containing X . Note that if \mathcal{C} is a permutation class then \mathcal{C} and $\mathcal{W}(\mathcal{C})$ contain the same simple permutations.

Direct sums, skew sums, and sum closures. Two inflations in particular occur frequently enough that they deserve their own names: the *direct sum* $\pi \oplus \sigma = 12[\pi, \sigma]$ and the *skew sum* $\pi \ominus \sigma = 21[\pi, \sigma]$. A class \mathcal{C} is said to be *sum closed* if $\pi \oplus \sigma \in \mathcal{C}$ whenever $\pi, \sigma \in \mathcal{C}$. Analogously, a class is said to be *skew sum closed* if $\pi \ominus \sigma \in \mathcal{C}$ for all $\pi, \sigma \in \mathcal{C}$. Furthermore, a permutation is said to be *sum indecomposable* if it cannot be written as the direct sum of two shorter (but nonempty) permutations and *skew sum indecomposable* if it cannot be written as the skew sum of two shorter permutations.

Recall that the *permutation graph* corresponding to π of length n is the graph G_π with vertices labeled by $[n]$, where $i \sim j$ if $i < j$ and $\pi(i) > \pi(j)$, i.e., if the entries in positions i and j form an inversion. It is easy to see that this graph captures the notion of sum indecomposability:

Proposition 1.2. *The permutation π is sum indecomposable if and only if G_π is connected.*

Proof. Suppose π is of length n . Clearly the vertices of any connected component of G_π must be labeled by an interval $[i, j]$. Thus if G_π is disconnected, it contains a connected component of the form $[1, j]$ for some j , from which it follows that $\pi = \pi(1) \cdots \pi(j) \oplus \pi(j+1) \cdots \pi(n)$. On the other hand, if $\pi = \sigma \oplus \tau$ then G_π is the disjoint union of G_σ and G_τ , and thus disconnected. \square

In fact, the condition of connectedness in Proposition 1.2 can be slightly weakened as shown by Proposition 6.2.

Given a set of permutations X we define the *sum completion* of X , denoted by $\bigoplus X$, to be the smallest sum complete permutation class containing X . Equivalently,

$$\bigoplus X = \{\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k : \text{each } \pi_i \text{ is contained in an element of } X\}.$$

When X is a singleton, $X = \{\pi\}$ say, we simply write $\bigoplus \pi$ for its sum completion. We analogously define the *skew sum completion* of X to be the smallest skew sum complete permutation class containing X . Counting sum complete classes can be quite easy, given enough information about the sum indecomposables:

Proposition 1.3. *Let f denote the generating function for the set of sum indecomposable permutations contained in members of X (not counting the empty permutation). Then the generating function for $\bigoplus X$ is $1/(1-f)$.*

Proof. There is a canonical bijection between elements of $\bigoplus X$ and sequences of nonempty sum indecomposable permutations in X . Therefore the generating function for $\bigoplus X$ is $1 + f + f^2 + \cdots$, establishing the proposition. \square

As our interest lies in growth rates rather than exact enumeration, we typically follow a use of Proposition 1.3 with an application of Pringsheim's Theorem:

Pringsheim's Theorem (see Flajolet and Sedgewick [14, Section IV.3]). *The upper growth rate, $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n}$, of a sequence $(a_n)_{n \geq 0}$ of nonnegative numbers is equal to the reciprocal of the smallest positive pole of the power series $\sum_{n \geq 0} a_n x^n$.*

We conclude our discussion of sums and skew sums with the following result of Arratia.

Proposition 1.4 (Arratia [5]). *Every sum or skew sum closed permutation class has a (possibly infinite) growth rate.*

Proof. Suppose, without loss, that \mathcal{C} is a sum closed permutation class. Then \oplus gives an injection from $\mathcal{C}_m \times \mathcal{C}_n$ to \mathcal{C}_{m+n} , so the sequence $|\mathcal{C}_n|$ is supermultiplicative and thus $\lim_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}$ exists by Fekete's Lemma². \square

Partial well-order. The poset (P, \leq) is said to be *partially well-ordered* (*pwo*) if it does not contain an infinite antichain. The results stated here have been rediscovered many times and so we will not attempt attribution.

Proposition 1.5. *For a poset (P, \leq) without infinite descending chains — and thus in particular, for any permutation class — the following are equivalent:*

- (1) P is pwo,
- (2) P contains no infinite antichain,
- (3) every infinite sequence of elements of P contains an infinite ascending sequence.

Given posets $(P_1, \leq_1), \dots, (P_s, \leq_s)$, the product order $(P_1, \leq_1) \times \dots \times (P_s, \leq_s)$ is the poset containing the tuples $P_1 \times \dots \times P_s$, equipped with the order $(x_1, \dots, x_s) \leq (y_1, \dots, y_s)$ if and only if $x_i \leq_i y_i$ for all $i \in [s]$.

Proposition 1.6. *The product $(P_1, \leq_1) \times \dots \times (P_s, \leq_s)$ of a collection of partial orders is pwo if and only if each of them is pwo.*

We now specialize to permutation classes.

Proposition 1.7. *The subclasses of a pwo permutation class satisfy the descending chain condition, i.e., if \mathcal{C} is a pwo class, there does not exist a sequence $\mathcal{C} = \mathcal{C}^0 \supsetneq \mathcal{C}^1 \supsetneq \mathcal{C}^2 \supsetneq \dots$ of permutation classes, or in other words, the subclasses of \mathcal{C} are well-founded under \subset .*

²Fekete's Lemma in its more typical form says that if a_n is superadditive, meaning that $a_{m+n} \geq a_m + a_n$, then $\lim_{n \rightarrow \infty} a_n/n$ exists and is equal to $\sup a_n/n$. To apply this form of Fekete's Lemma in our context, consider the sequence $\log |\mathcal{C}_n|$.

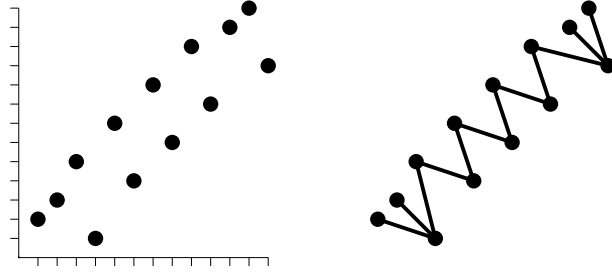


Figure 2: The plot (left) and permutation graph (right) of u_4 .

Proof. Suppose to the contrary that \mathcal{C} contains an infinite strictly decreasing sequence $\mathcal{C} = \mathcal{C}^0 \supsetneq \mathcal{C}^1 \supsetneq \mathcal{C}^2 \supsetneq \dots$ and for each $i \geq 1$ choose $\beta_i \in \mathcal{C}^i \setminus \mathcal{C}^{i-1}$. As each β_i lies in the pwo class \mathcal{C} , Proposition 1.5 (3) shows that the sequence β_1, β_2, \dots contains an ascent, say $\beta_i \leq \beta_j$ for some indices $i < j$. This, however, shows that $\beta_j \notin \mathcal{C}^i$, contradicting our choice of β_j . \square

Proposition 1.8. *A pwo permutation class contains only countably many subclasses.*

Proof. Let \mathcal{C} be a pwo permutation class. Every subclass $\mathcal{D} \subseteq \mathcal{C}$ is of the form $\mathcal{C} \cap \text{Av}(B)$ for some antichain $B \subseteq \mathcal{C}$. Since \mathcal{C} is pwo, all such antichains are finite and thus the set of subclasses of \mathcal{C} is countable. \square

Of particular importance to us is the connection between simple permutations and pwo permutation classes:

Proposition 1.9 (see Albert and Atkinson [1] or Gustedt [16]). *Every permutation class containing only finitely many simple permutations is pwo.*

One of the most straightforward infinite antichains to describe is $U = \{u_1, u_2, \dots\}$ defined by

$$\begin{aligned}
 u_1 &= 2, 3, 5, 1 \mid \mid 6, 7, 4 \\
 u_2 &= 2, 3, 5, 1 \mid 7, 4 \mid 8, 9, 6 \\
 u_3 &= 2, 3, 5, 1 \mid 7, 4, 9, 6 \mid 10, 11, 8 \\
 &\vdots \\
 u_k &= 2, 3, 5, 1 \mid 7, 4, 9, 6, 11, 8, \dots, 2k+3, 2k \mid 2k+4, 2k+5, 2k+2 \\
 &\vdots
 \end{aligned}$$

(Here the vertical bars have no mathematical meaning but are meant only to emphasize the different parts of the permutations.)

Proposition 1.10. *The set U forms an infinite antichain of permutations.*

Proof, due to Klazar [22]. Clearly the permutation graph G_σ must be contained, as an induced subgraph, in G_π whenever $\sigma \leq \pi$, and almost as clearly, the set of permutation graphs $\{G_{u_1}, G_{u_2}, \dots\}$ forms an infinite antichain under the induced subgraph order. \square

The antichain U leads to an upper bound for Answer 2, but first we must make the following observation.

Proposition 1.11. *Every permutation class containing an infinite antichain contains uncountably many subclasses.*

Proof. Suppose the class \mathcal{C} contains the infinite antichain A . Then every class of the form $\mathcal{C} \cap \text{Av}(B)$, $B \subseteq A$ is distinct. \square

Now note that

$$U \subseteq \text{Av}(321, 3412, 4123, 23451, 134526, 134625, 314526, 314625).$$

The Maple package FINLABEL, described in Vatter [28], computes the generating function of this class to be

$$\frac{x(1 + x + x^2 + 2x^3 + 3x^4 + 3x^5 + x^6 - x^7 - x^9)}{(1 + x)(1 - 2x - x^3)},$$

which shows (via Pringsheim's Theorem) that its growth rate is κ . Therefore there are uncountably many permutation classes with upper growth rate at most κ .

Atomicity. A permutation class is said to be *atomic* if it cannot be written as the union of two proper subclasses. Like partial well-order, atomicity (under a variety of names) has been rediscovered numerous times; it seems to date originally to the 1954 article Fraïssé [15]. Murphy undertook a particularly thorough investigation of atomic permutation classes in his thesis [24], and all of the results here can be found there.

It is not difficult to show that the *joint embedding property* is a necessary and sufficient condition for the permutation class \mathcal{C} to be atomic; this condition states that for all $\pi, \sigma \in \mathcal{C}$, there is a $\tau \in \mathcal{C}$ containing both π and σ . Fraïssé established another necessary and sufficient condition for atomicity, which we describe only in the permutation context (Fraïssé proved his results for relational structures). Given two linearly ordered sets (or simply, linear orders) A and B and a bijection $f : A \rightarrow B$, every finite subset $\{a_1 < \dots < a_n\} \subseteq A$ maps to a finite sequence $f(a_1), \dots, f(a_n) \in B$ that is order isomorphic to a unique permutation. We call the set of permutations that arise in this manner the *age* of f , denoted $\text{Age}(f : A \rightarrow B)$.

Theorem 1.12 (Fraïssé [15]; see also Hodges [17, Section 7.1]). *For a permutation class \mathcal{C} , the following are equivalent:*

- (1) \mathcal{C} is atomic,
- (2) \mathcal{C} satisfies the joint embedding property, and

(3) $\mathcal{C} = \text{Age}(f : A \rightarrow B)$ for a bijection f between two countable linear orders A and B .

We use only the generic properties of atomic classes, i.e., those that hold for any appropriate type of object, but note that atomic classes of permutations are particularly interesting³.

In addition to Theorem 1.12, we need several results about atomicity for pwo classes. For the first, we follow the proof given by Murphy [24].

Proposition 1.13. *Every pwo permutation class can be written as a finite union of atomic classes.*

Proof. Consider the binary tree whose root is the pwo class \mathcal{C} , all of whose leaves are atomic classes, and in which the children of the non-atomic class \mathcal{D} are two proper subclasses $\mathcal{D}^1, \mathcal{D}^2 \subsetneq \mathcal{D}$ such that $\mathcal{D}^1 \cup \mathcal{D}^2 = \mathcal{D}$. Because \mathcal{C} is pwo its subclasses satisfy by the descending chain condition by Proposition 1.7, so this tree contains no infinite paths and thus is finite; its leaves give the desired atomic classes. \square

The problem of computing growth rates of pwo classes can then be reduced to that of computing growth rates of atomic classes:

Proposition 1.14. *For a pwo permutation class \mathcal{C} , $\overline{\text{gr}}(\mathcal{C})$ is equal to the maximum upper growth rate of an atomic subclass of \mathcal{C} .*

Proof. Using Proposition 1.13, write \mathcal{C} as a union of finitely many atomic subclasses, $\mathcal{C} = \mathcal{C}^1 \cup \dots \cup \mathcal{C}^m$, and then choose an infinite subsequence $n_1 < n_2 < \dots$ such that $\overline{\text{gr}}(\mathcal{C}) = \lim_{i \rightarrow \infty} \sqrt[n_i]{|\mathcal{C}_{n_i}|}$. For each n , at least $1/m$ of the permutations in \mathcal{C}_n lie in a particular \mathcal{C}^j , and thus there is an infinite subsequence of the n_i s, say $n'_1 < n'_2 < \dots$, such that at least $1/m$ of the permutations in $\mathcal{C}_{n'_i}$ lie in the same \mathcal{C}^j for all i . Then

$$\overline{\text{gr}}(\mathcal{C}) = \lim_{i \rightarrow \infty} \sqrt[n_i]{|\mathcal{C}_{n_i}|} = \lim_{i \rightarrow \infty} \sqrt[n'_i]{|\mathcal{C}_{n'_i}|} \leq \limsup_{i \rightarrow \infty} \sqrt[n'_i]{m|\mathcal{C}_{n'_i}^j|} \leq \overline{\text{gr}}(\mathcal{C}^j).$$

The reverse inequality is obvious, proving the proposition. \square

2. GENERALIZED GRID CLASSES

When discussing specific grid classes (which will rarely be necessary herein), we index matrices beginning from the lower left-hand corner and we reverse the rows and columns,

³ For instance, define $\mathcal{T}(X, Y)$ as the set of all (necessarily atomic) classes of permutations that can be expressed as $\text{Age}(f : X \rightarrow Y)$. The following two questions then naturally arise:

- For a given X and Y , can one characterize $\mathcal{T}(X, Y)$? Can it be decided whether a given class lies in $\mathcal{T}(X, Y)$?
- For what linear orders X, Y, W , and Z is $\mathcal{T}(X, Y) \subseteq \mathcal{T}(W, Z)$?

For some partial answers to these questions, the reader is referred to Atkinson, Murphy, and Ruškuc [7] and Huczynska and Ruškuc [19].

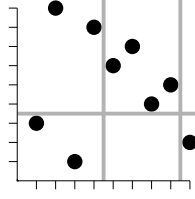


Figure 3: An $\begin{pmatrix} \text{Av}(12) & \text{Av}(321) & \emptyset \\ \text{Av}(12) & \emptyset & \{1\} \end{pmatrix}$ -gridding of the permutation 391867452; here the column divisions are $c_1, c_2, c_3, c_4 = 1, 5, 9, 10$ and the row divisions are $r_1, r_2, r_3 = 1, 4, 10$.

so $\mathcal{M}_{3,2}$ denotes for us the entry of \mathcal{M} in the 3rd column from the left and 2nd row from the bottom. Below we include a matrix with its entries labeled:

$$\begin{pmatrix} (1, 2) & (2, 2) & (3, 2) \\ (1, 1) & (2, 1) & (3, 1) \end{pmatrix}.$$

Roughly, the grid class of a matrix \mathcal{M} is the set of all permutations that can be divided into a finite number of blocks, each containing a subsequence of a prescribed form, also dictated by \mathcal{M} . There are some notational prerequisites to be covered before they can be defined formally.

Given a permutation π of length n and sets $X, Y \subseteq [n]$, we write $\pi(X \times Y)$ for the permutation that is order isomorphic to the subsequence of π with indices from X which has values in Y . For example, to compute $391867452([3, 7] \times [2, 6])$ we consider the subsequence of entries in indices 3 through 7, 18674, which have values between 2 and 6; in this case the subsequence is 64, so $391867452([3, 7] \times [2, 6]) = 21$.

Suppose that \mathcal{M} is a $t \times u$ matrix (meaning, in our indexing, that \mathcal{M} has t columns and u rows) whose entries are permutation classes. An \mathcal{M} -gridding of the permutation π of length n is a pair of sequences $1 = c_1 \leq \dots \leq c_{t+1} = n + 1$ (the column divisions) and $1 = r_1 \leq \dots \leq r_{u+1} = n + 1$ (the row divisions) such that $\pi([c_k, c_{k+1}) \times [r_\ell, r_{\ell+1}))$ is a member of $\mathcal{M}_{k,\ell}$ for all $k \in [t]$ and $\ell \in [u]$. Figure 3 shows an example.

The *grid class* of \mathcal{M} , written $\text{Grid}(\mathcal{M})$, consists of all permutations which possess an \mathcal{M} -gridding. Furthermore, we say that the permutation class \mathcal{C} is itself \mathcal{M} -griddable if $\mathcal{C} \subseteq \text{Grid}(\mathcal{M})$. Note that our treatment of griddability is more general than earlier definitions from [20, 25, 30] which consider only *monotone grid classes*, defined as classes of the form $\text{Grid}(\mathcal{M})$ where each entry of \mathcal{M} is $\text{Av}(12)$, $\text{Av}(21)$, or the empty class \emptyset .

We sometimes need to consider particular griddings of permutations, and in this case refer to a permutation together with an \mathcal{M} -gridding (if it has one) as an \mathcal{M} -gridded permutation (as opposed to an \mathcal{M} -griddable permutation); note that the set of all \mathcal{M} -gridded permutations will generally contain many different \mathcal{M} -griddings of the same permutation. However, as demonstrated by the next proposition, this does not affect the logarithmic asymptotics we are concerned with.

Proposition 2.1. *For a matrix \mathcal{M} of permutation classes and an \mathcal{M} -griddable class \mathcal{C} , the upper (resp., lower) growth rate of \mathcal{C} is equal to the upper (resp., lower) growth rate of the sequence enumerating the \mathcal{M} -gridded permutations in \mathcal{C} .*

Proof. Suppose that \mathcal{M} is of dimensions $t \times u$ and let g_n denote the number of \mathcal{M} -gridded permutations of length n in \mathcal{C} . As every permutation in \mathcal{C} has at least one \mathcal{M} -gridding (\mathcal{C} is \mathcal{M} -griddable) and no permutation of length n possesses more than $\binom{n+t}{t} \binom{n+u}{u}$ different \mathcal{M} -griddings (the column divisions form a multiset of size t chosen from the set $[n+1]$, the row divisions, a multiset of size u) we get that

$$g_n / \binom{n+t}{t} \binom{n+u}{u} \leq |\mathcal{C}_n| \leq g_n,$$

from which the proposition immediately follows. \square

We say that the class \mathcal{C} is $\{\mathcal{D}^1, \dots, \mathcal{D}^m\}$ -griddable if \mathcal{C} is \mathcal{M} -griddable for some matrix \mathcal{M} whose entries are all either empty or subclasses of one of the \mathcal{D}^i s. In the $m = 1$ case we abbreviate “ $\{\mathcal{D}\}$ -griddable” to “ \mathcal{D} -griddable”, a situation which, as the following proposition shows, is no less general.

Proposition 2.2. *A permutation class is $\{\mathcal{D}^1, \dots, \mathcal{D}^m\}$ -griddable if and only if it is $\mathcal{D}^1 \cup \dots \cup \mathcal{D}^m$ -griddable.*

Proof. If the class \mathcal{C} is $\{\mathcal{D}^1, \dots, \mathcal{D}^m\}$ -griddable then it is clearly $\mathcal{D}^1 \cup \dots \cup \mathcal{D}^m$ -griddable. For the other direction, suppose that \mathcal{C} is \mathcal{M} -griddable for a $t \times u$ matrix \mathcal{M} whose entries are all (without loss) equal to $\mathcal{D}^1 \cup \dots \cup \mathcal{D}^m$. Thus for every cell (j, k) , the entries in cell (j, k) in any \mathcal{M} -gridding of any permutation $\pi \in \mathcal{C}$ lie in one of the classes \mathcal{D}^i . Therefore every $\pi \in \mathcal{C}$ is \mathcal{N} -griddable for the matrix \mathcal{N} which consists of the direct sum⁴ of every one of the (finitely many) $t \times u$ matrices with entries from $\{\mathcal{D}^1, \dots, \mathcal{D}^m\}$. \square

It is useful to have several different perspectives of griddability, which require a bit more notation. Given a permutation class \mathcal{D} , we say that the permutation π of length n can be covered by s \mathcal{D} -rectangles if there are (not necessarily disjoint) rectangles $[w_1, x_1] \times [y_1, z_1], \dots, [w_s, x_s] \times [y_s, z_s] \subseteq [n] \times [n]$ such that

- for each $i \in [s]$, $\pi([w_i, x_i] \times [y_i, z_i]) \in \mathcal{D}$, and
- for each $i \in [n]$, the point $(i, \pi(i))$ lies in $\bigcup_{i \in [s]} [w_i, x_i] \times [y_i, z_i]$.

For the third perspective, we say that the line L slices the rectangle R if L intersects the interior of R . If \mathcal{R} is a collection of rectangles and \mathcal{L} a collection of lines, we say that \mathcal{L} slices \mathcal{R} if every rectangle in \mathcal{R} is sliced by some line from \mathcal{L} .

⁴The direct sum of the matrices A and B is defined as $\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$.

Proposition 2.3. *For permutation classes \mathcal{C} and \mathcal{D} the following are equivalent:*

- (1) \mathcal{C} is \mathcal{D} -griddable,
- (2) *there is a constant s so that for every permutation $\pi \in \mathcal{C}$, the set $\{\text{axes-parallel rectangles } R : \pi(R) \notin \mathcal{D}\}$ can be sliced by a collection of s horizontal and vertical lines, and*
- (3) *there is a constant s so that every permutation in \mathcal{C} can be covered by s \mathcal{D} -rectangles.*

Proof. To begin with, if \mathcal{C} is \mathcal{D} -griddable then \mathcal{C} is \mathcal{M} -griddable for a matrix \mathcal{M} of some dimensions, say $t \times u$, whose entries consist of subclasses of \mathcal{D} . Thus for every permutation $\pi \in \mathcal{C}_n$ there are column and row divisions $1 = c_1 \leq \dots \leq c_{t+1} = n + 1$ and $1 = r_1 \leq \dots \leq r_{u+1} = n + 1$ so that every subpermutation $\pi([c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}])$ lies in \mathcal{D} . Therefore the corresponding lines, $x = c_1, \dots, x = c_t, y = r_1, \dots, y = r_u$, slice the given collection of rectangles, verifying that (1) implies (2). That (2) implies (3) is similarly clear: any such collection of lines will slice the plane into a collection of rectangles which contain points order isomorphic to elements of \mathcal{D} .

This leaves us to establish that (3) implies (1). Suppose that the permutation π of length n is covered by the s \mathcal{D} -rectangles $[w_1, x_1] \times [y_1, z_1], \dots, [w_s, x_s] \times [y_s, z_s] \subseteq [n] \times [n]$. Define the indices c_1, \dots, c_{2s} and r_1, \dots, r_{2s} by

$$\begin{aligned} \{c_1 \leq \dots \leq c_{2s}\} &= \{w_1, x_1, \dots, w_s, x_s\}, \\ \{r_1 \leq \dots \leq r_{2s}\} &= \{y_1, z_1, \dots, y_s, z_s\}. \end{aligned}$$

Since these rectangles cover π , we must have $c_1 = r_1 = 1$ and $c_{2s} = r_{2s} = n$. Now we claim that these sets of indices give a $(2s - 1) \times (2s - 1)$ \mathcal{D} -gridding of π .

To prove this claim it suffices to show that $\pi([c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}]) \in \mathcal{D}$ for every $k, \ell \in [2s - 1]$. Because the rectangles given cover π , the point (c_k, r_ℓ) lies in at least one rectangle, say $[w_m, x_m] \times [y_m, z_m]$. Thus $c_k \geq w_m$ and $r_\ell \geq y_m$ and, because of the ordering of the c s and r s, we have $c_{k+1} \leq x_m$ and $r_{\ell+1} \leq z_m$. Therefore $[c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}]$ is contained in $[w_m, x_m] \times [y_m, z_m]$ and so $\pi([c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}]) \in \mathcal{D}$.

Therefore, if (2) holds, then the above argument implies that every permutation in \mathcal{C} is \mathcal{M} -griddable for the $(2s - 1) \times (2s - 1)$ matrix \mathcal{M} whose every entry is \mathcal{D} , and thus \mathcal{C} is \mathcal{D} -griddable, as desired. \square

This language makes the following already fairly obvious fact a bit easier to prove.

Proposition 2.4. *If \mathcal{C} is \mathcal{D} -griddable and \mathcal{D} is \mathcal{E} -griddable then \mathcal{C} is \mathcal{E} -griddable.*

Proof. Since \mathcal{C} is \mathcal{D} -griddable, Proposition 2.3 shows that there is a constant s so that every permutation in \mathcal{C} can be covered by s \mathcal{D} -rectangles. Similarly, there is a constant t so that every permutation in \mathcal{D} , and thus every \mathcal{D} -rectangle, can be covered by t \mathcal{E} -rectangles. This shows that every permutation in \mathcal{C} can be covered by st \mathcal{E} -rectangles, which, by Proposition 2.3, establishes the proposition. \square

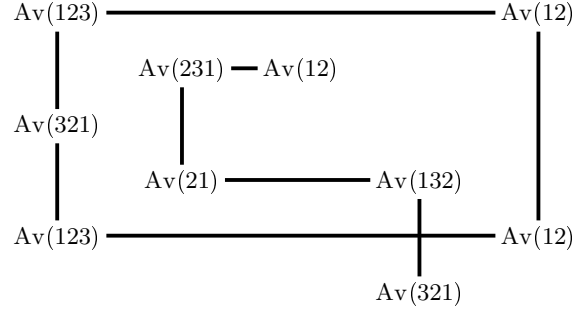


Figure 4: The graph of a matrix of permutation classes.

We conclude this section by discussing graphs of grid classes and their pwo and growth rate properties. The *graph of the matrix \mathcal{M} of permutation classes*, $G_{\mathcal{M}}$, is for us the graph on the vertices $\{(i, j) : \mathcal{M}_{i,j} \neq \emptyset\}$ in which $(i, j) \sim (k, \ell)$ if (i, j) and (k, ℓ) share either a row or a column, and there are no nonempty cells between them in this row or column. Further, we label the vertex (i, j) in this graph by the class it corresponds to, $\mathcal{M}_{i,j}$. Figure 4 shows the graph of a matrix. (This graph is a slight deviation from the graphs studied in [20] and [25], but these differences do not affect the results stated here.) The main result of [25] can be stated in this language as:

Theorem 2.5 (Murphy and Vatter [25]). *If \mathcal{M} contains only monotone and empty classes then $\text{Grid}(\mathcal{M})$ is pwo if and only if $G_{\mathcal{M}}$ is a forest.*

A much simpler proof that $\text{Grid}(\mathcal{M})$ is pwo whenever $G_{\mathcal{M}}$ is a forest of monotone classes can be found in Waton’s thesis [30]; in that thesis Waton also characterizes the atomic monotone grid classes, a characterization which has a natural graph-theoretic statement as well.

We also need a coarsening of the permutation containment order; for two \mathcal{M} -gridded permutations σ and π of respective lengths k and n , we say that π *contains a gridded copy of σ* and write $\sigma \leq_g \pi$ if there are indices $1 \leq i_1 < \dots < i_k \leq n$ such that the subsequence $\pi(i_1) \dots \pi(i_k)$ is order isomorphic to σ (this is the normal permutation containment order) and, further, that for every $j \in [k]$, $\sigma(j)$ and $\pi(i_j)$ lie in the same cell of \mathcal{M} in the accompanying \mathcal{M} -griddings of σ and π . This is indeed coarser than the normal ordering because $\sigma \leq_g \pi \implies \sigma \leq \pi$.

Finally, we define a *connected component* of the matrix \mathcal{M} of permutation classes to be a submatrix of \mathcal{M} whose cells give rise to a connected component of $G_{\mathcal{M}}$. Further, if A is any subset of cells of \mathcal{M} then we write \mathcal{M}^A to denote the submatrix of \mathcal{M} formed by the cells in A . For example, the matrix \mathcal{M} whose graph $G_{\mathcal{M}}$ is depicted in Figure 4 contains two connected components:

$$\mathcal{M}^{\{(1,2),(1,4),(1,6),(5,6),(5,2)\}} = \begin{pmatrix} \text{Av}(123) & \text{Av}(12) \\ \text{Av}(321) & \\ \text{Av}(123) & \text{Av}(12) \end{pmatrix}$$

and

$$\mathcal{M}^{\{(4,1),(4,3),(2,3),(2,5),(3,5)\}} = \begin{pmatrix} \text{Av}(231) & \text{Av}(12) & \\ \text{Av}(21) & & \text{Av}(132) \\ & & \text{Av}(321) \end{pmatrix}.$$

Proposition 2.6. *If the grid classes of each of its connected components are pwo then $\text{Grid}(\mathcal{M})$ is pwo.*

Proof. Label the connected components of \mathcal{M} by A_1, \dots, A_s , viewing each A_i as a subset of cells of \mathcal{M} . There is a canonical order preserving bijection between the poset of all \mathcal{M} -gridded permutations, ordered by \leq_g , and the poset of tuples (π^1, \dots, π^s) where each π^i is an \mathcal{M}^{A_i} -gridded permutation, ordered by the product order $\leq_g \times \dots \times \leq_g$. The proof is then completed by Proposition 1.6. \square

We need a bit more notation for the last proposition of the section. For an \mathcal{M} -gridded permutation π and a subset $A = \{(j_1, k_1), \dots, (j_s, k_s)\}$ of cells of \mathcal{M} , we say that the (gridded) permutation formed by the entries of π lying in the cells of A is the *restriction of the gridded permutation π to A* . (Note that different griddings of π will tend to lead to different restrictions.) Furthermore, if \mathcal{C} is a class, we say that \mathcal{C} 's *restriction to A* is the set of all restrictions of its members to A . Note that the restriction of a permutation class to a set of cells gives a set of gridded permutations which is closed downward under \leq_g .

Proposition 2.7. *Suppose that \mathcal{C} is \mathcal{M} -griddable. The upper growth rate of \mathcal{C} is the maximum of the upper growth rates of its restrictions to connected components of \mathcal{M} .*

Proof. Let g_n denote the number of \mathcal{M} -gridded length n permutation in \mathcal{C} , so the upper growth rate of \mathcal{C} is equal to the upper growth rate of g_n by Proposition 2.1. Now label the connected components of \mathcal{M} by A_1, \dots, A_s , denote the restriction of \mathcal{C} to A_i by \mathcal{C}^i , and suppose that the greatest upper growth rate of any of these restrictions is γ . It suffices to establish that the upper growth rate of g_n is at most γ .

As remarked in the proof of Proposition 2.6, there is a canonical bijection between \mathcal{M} -gridded permutations and the poset of tuples (π^1, \dots, π^s) where each π^i is an \mathcal{M}^{A_i} -gridded permutation. Letting $g_{i,n}$ denote the number of \mathcal{M}^{A_i} -gridded permutations of length n in \mathcal{C}^i we have

$$g_n \leq \sum_{n_1 + \dots + n_s = n} \prod_i g_{i,n_i}.$$

Now fix $\epsilon > 0$. By our choice of γ , there is some N such that for all $i \in [s]$ and $n_i > N$, $g_{i,n_i} < ((1 + \epsilon)\gamma)^{n_i}$. There is also (trivially) an integer $N' \geq N$ such that for all $i \in [s]$, $n > N'$, and $n_i \leq N$, $g_{i,n_i} \leq (1 + \epsilon)^n$. Thus we have that for all $n > N'$ and $n_i \leq n$,

$$g_{i,n_i} \leq \begin{cases} (1 + \epsilon)^n \gamma^{n_i} & \text{if } n_i > N \text{ while} \\ (1 + \epsilon)^n & \text{if } n_i \leq N. \end{cases}$$

It follows that for these values of n ,

$$\begin{aligned} g_n &\leq \binom{n+s-1}{s-1} \max_{n_1+\dots+n_s=n} \prod_i g_{i,n_i}, \\ &\leq \binom{n+s-1}{s-1} \max_{n_1+\dots+n_s=n} (1+\epsilon)^{ns} \gamma^n, \end{aligned}$$

implying that $\limsup_{n \rightarrow \infty} \sqrt[n]{g_n} \leq (1+\epsilon)^s \gamma$. Letting $\epsilon \rightarrow 0$ completes the proof. \square

3. CHARACTERIZING \mathcal{D} -GRIDDABLE AND GRID IRREDUCIBLE CLASSES

We begin this section with a characterization:

Theorem 3.1. *The permutation class \mathcal{C} is \mathcal{D} -griddable if and only if it does not contain arbitrarily long sums or skew sums of basis elements of \mathcal{D} , that is, if there exists a constant m so that \mathcal{C} does not contain $\beta_1 \oplus \dots \oplus \beta_m$ or $\beta_1 \ominus \dots \ominus \beta_m$ for any basis elements β_1, \dots, β_m of \mathcal{D} .*

N.b. If \mathcal{D} is finitely based then this condition can be simplified: \mathcal{C} fails to have a \mathcal{D} -gridding if and only if \mathcal{C} contains $\oplus \beta$ or $\ominus \beta$ for some basis element β of \mathcal{D} .

One direction of Theorem 3.1 is clear: if β_1, \dots, β_m are basis elements of \mathcal{D} then their direct sum $\beta_1 \oplus \dots \oplus \beta_m$ can be covered by no fewer than $m+1$ \mathcal{D} -rectangles, so if \mathcal{C} contains arbitrarily long direct sums (equivalently, skew sums) of basis elements of \mathcal{D} then it is not \mathcal{D} -griddable.

By the Proposition 2.3 (2) interpretation of griddability, the other direction of Theorem 3.1 involves slicing a collection of rectangles in the plane — in particular, the set $\{\text{axes-parallel rectangles } R : \pi(R) \notin \mathcal{D}\}$ — with a bounded number of vertical and horizontal lines. Since we use these notions again in the next section, we cast this discussion in slightly more general terms.

We say that two rectangles R, S are *independent* if both their x - and y -axis projections are disjoint, and a set of rectangles is said to be independent if they are pairwise independent. An *increasing set* of rectangles is an independent set of rectangles $\{R_1, \dots, R_m\}$ such that R_2 lies above and to the right of R_1 , R_3 lies above and to the right of R_2 , and so on. *Decreasing sets* of rectangles are defined analogously. Note that independent sets of rectangles can (essentially) be viewed as permutations; thus they fall under the purview of the Erdős-Szekeres Theorem, so every independent set of $(m-1)^2 + 1$ rectangles contains either an increasing or a decreasing subset of m rectangles.

Returning to the context of Theorem 3.1, if the class \mathcal{C} satisfies the hypotheses of that theorem then for any permutation $\pi \in \mathcal{C}$ the set $\mathcal{R} = \{\text{axes-parallel rectangles } R : \pi(R) \notin \mathcal{D}\}$ does not contain an increasing or a decreasing set of m rectangles (if it did, then each such rectangle would contain a basis element of \mathcal{D} , and thus π , and therefore \mathcal{C} , would contain $\beta_1 \oplus \dots \oplus \beta_m$ or $\beta_1 \ominus \dots \ominus \beta_m$ for some basis elements β_1, \dots, β_m of \mathcal{D}). The proof of the theorem is therefore completed with the following lemma.

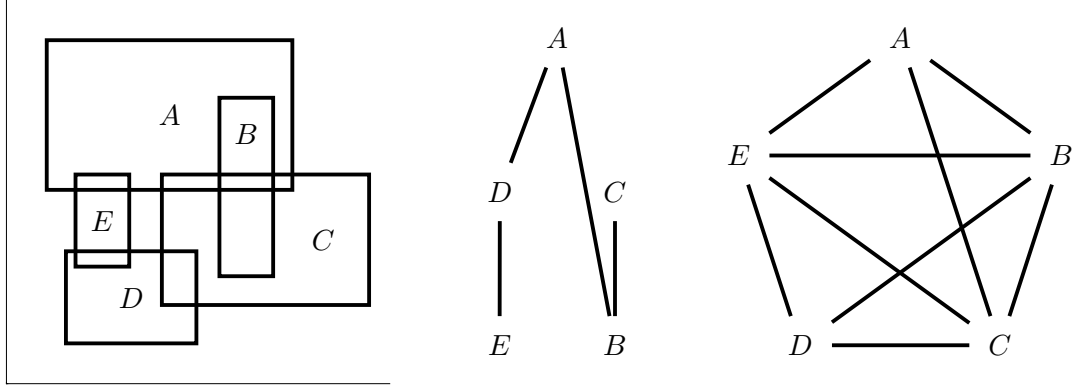


Figure 5: A collection of axes-parallel rectangles (left) with their \subseteq_x quasi-order (center — in this case it happens to be a partial order) and \sim_y graph (right).

Lemma 3.2. *There is a function $f(m)$ such that for any collection \mathcal{R} of axes-parallel rectangles in the plane which has no independent set of size m or greater, there exists a set of $f(m)$ horizontal and vertical lines that slice every rectangle in \mathcal{R} .*

Proof. Our proof is by induction on m ; note that the base case $m = 0$ is trivial. We denote by $\text{proj}_x R$ and $\text{proj}_y R$ the projections of R onto the x - and y -axes, respectively, and define two structures on the set \mathcal{R} . The first is a quasi-order, $R \subseteq_x S$ if $\text{proj}_x R \subseteq \text{proj}_x S$, while the second is a graph, $R \sim_y S$ if $\text{proj}_y R \cap \text{proj}_y S \neq \emptyset$. Figure 5 shows an example.

Consider first a clique, say $\mathcal{K} \subseteq \mathcal{R}$, in the \sim_y graph. Every pair of rectangles in this clique have intersecting y -projections, and thus it is easy to see (this is the one-dimensional version of Helly's Theorem) that $\bigcap_{R \in \mathcal{K}} \text{proj}_y R \neq \emptyset$. Hence every clique in the \sim_y graph can be sliced by a single horizontal line. In particular, if the \sim_y graph is complete then \mathcal{R} can be sliced by a single line.

We therefore assume that \sim_y is not complete. Choose $R_1 \not\sim_y R_2$ from \mathcal{R} to minimize $\max(\text{ht}(R_1), \text{ht}(R_2))$, and without loss suppose that $\text{ht}(R_1) \leq \text{ht}(R_2)$ and that R_1 lies below R_2 . Therefore $\{R \in \mathcal{R} : \text{ht}(R) < \text{ht}(R_2)\}$ forms a clique in the \sim_y graph, and thus by the above, these rectangles can be sliced with a single horizontal line.

Now consider the regions shown in Figure 6. By induction and our choice of R_1 and R_2 , we have the following:

- (a) The rectangles completely contained in this region, which consists of all points strictly below and to the left of R_2 , cannot contain an independent set of $m-1$ or more rectangles, as otherwise R_2 together with this independent set would form an independent set of size m . Thus this region can be sliced by $f(m-1)$ horizontal and vertical lines by induction.
- (b) Every rectangle completely contained in this region, which consists of all points strictly below R_2 whose x -coordinate lies strictly between the left and right edges

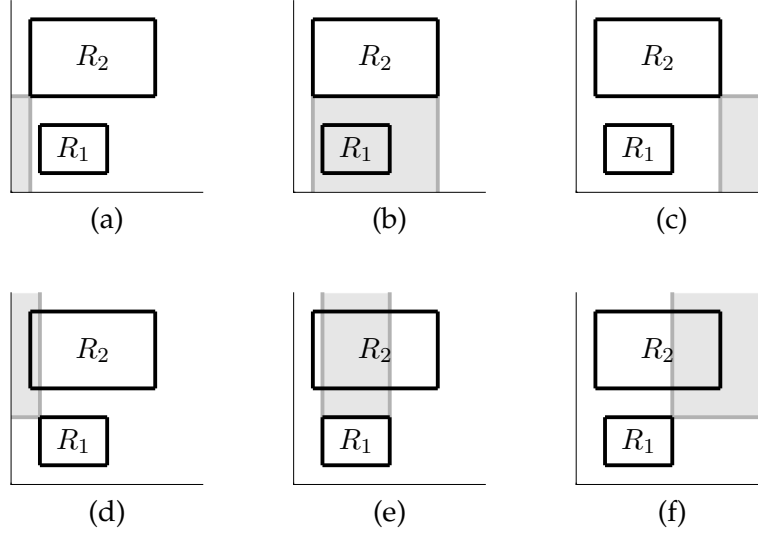


Figure 6: The six regions from the proof of Lemma 3.2.

of R_2 , has height strictly less than R_2 in the \subseteq_x quasi-order, and thus by our choice of R_2 these rectangles form a clique in the \sim_y graph and so can be sliced by a single horizontal line.

- (c) As with (a), these rectangles can be sliced by $f(m - 1)$ horizontal and vertical lines.
- (d) As with (a), these rectangles can be sliced by $f(m - 1)$ horizontal and vertical lines.
- (e) As with (b), these rectangles can be sliced by a single horizontal line.
- (f) As with (a), these rectangles can be sliced by $f(m - 1)$ horizontal and vertical lines.

Thus we have found a collection of $4f(m - 1) + 2$ vertical and horizontal lines which slice every rectangle properly contained in one of the regions (a)–(f). Furthermore, the 8 lines which coincide with the sides of R_1 and R_2 slice every rectangle that is not properly contained in one of these regions, and so we may take $f(m) = 4f(m - 1) + 10$, completing the proof. \square

A special case of Theorem 3.1, the characterization of monotone griddable classes, appeared in Huczynska and Vatter [20]. Note that the condition given there — a permutation class is monotone griddable if and only if it does not contain arbitrarily long sums of 21 or skew sums of 12 — is merely a simplification of the conditions given by Theorem 3.1⁵.

⁵*Proof.* Take \mathcal{D} to contain the monotone permutations, i.e., $\mathcal{D} = \text{Av}(12) \cup \text{Av}(21)$, and note that a class is \mathcal{D} -griddable if and only if it is monotone griddable by Proposition 2.2. The basis of \mathcal{D} is $\{132, 213, 231, 312\}$, so Theorem 3.1 implies that the permutation class \mathcal{C} is \mathcal{D} -griddable if and only if it does not contain arbitrarily long sums or skew sums of any of these elements, a condition equivalent to not containing arbitrarily long sums of 21 or skew sums of 12. \square

Moving beyond griddability, let us say that the class \mathcal{D} is *grid irreducible* if \mathcal{D} is not \mathcal{E} -griddable for any proper subclass $\mathcal{E} \subsetneq \mathcal{D}$. Theorem 3.1 gives us, almost immediately, a characterization of the grid irreducible classes.

Proposition 3.3. *The permutation class \mathcal{D} is grid irreducible if and only if $\mathcal{D} = \{1\}$ or, for every $\pi \in \mathcal{D}$, either $\oplus \pi$ or $\ominus \pi$ is contained in \mathcal{D} .*

Proof. If \mathcal{D} is finite then it is clear that \mathcal{D} is grid irreducible if and only if $\mathcal{D} = \{1\}$, so let us suppose that \mathcal{D} is infinite. If there is some $\pi \in \mathcal{D}$ such that neither $\oplus \pi$ nor $\ominus \pi$ is contained in \mathcal{D} then Theorem 3.1 shows that \mathcal{D} is $\mathcal{D} \cap \text{Av}(\pi)$ -griddable, and thus \mathcal{D} is not grid irreducible. If, on the other hand, \mathcal{D} contains arbitrarily long direct sums or skew sums of each of its members then it is clear that \mathcal{D} is not \mathcal{E} -griddable for any proper subclass $\mathcal{E} \subsetneq \mathcal{D}$. \square

Note that Proposition 3.3 does not guarantee that every class \mathcal{C} is \mathcal{D} -griddable for a grid irreducible subclass $\mathcal{D} \subseteq \mathcal{C}$, and indeed, this does not necessarily hold⁶; however, it is true in an important special case.

Proposition 3.4. *If the infinite permutation class \mathcal{C} is pwo then \mathcal{C} is \mathcal{D} -griddable for the grid irreducible subclass $\mathcal{D} = \{\pi \in \mathcal{C} : \text{either } \oplus \pi \text{ or } \ominus \pi \text{ is contained in } \mathcal{C}\}$.*

Proof. Set

$$R = \mathcal{C} \setminus \mathcal{D} = \{\pi \in \mathcal{C} : \text{neither } \oplus \pi \text{ nor } \ominus \pi \text{ is contained in } \mathcal{C}\}$$

and let B denote the minimal elements of R under the permutation containment order, so $\mathcal{D} = \mathcal{C} \cap \text{Av}(B)$. Since B forms an antichain and \mathcal{C} is pwo, B is finite, and so there is a least integer, say m , so that neither $\oplus^m \beta$ nor $\ominus^m \beta$ lies in \mathcal{C} for any $\beta \in B$.

By Theorem 3.1, we need to show that \mathcal{C} does not contain arbitrarily long sums or skew sums of basis elements of \mathcal{D} . Consider a sum $\sigma = \beta_1 \oplus \cdots \oplus \beta_{m|B|}$ of basis elements of \mathcal{D} . Suppose to the contrary that σ does lie in \mathcal{C} , so each β_i must contain an element of B . By the Pigeonhole Principle, at least m of the β_i s contain the same element of B , so σ contains $\oplus^m \beta$ for some $\beta \in B$, and thus $\sigma \notin \mathcal{C}$. The situation for skew sums is analogous. \square

We conclude this section with the following result; the inference important to us is that the combination of atomicity and grid irreducibility is quite strong, which is needed for the proof of Theorem 6.1.

Proposition 3.5. *The following conditions on a permutation class \mathcal{C} are equivalent:*

- (1) \mathcal{C} is both atomic and grid irreducible,
- (2) for every $\pi, \sigma \in \mathcal{C}$, either $\pi \oplus \sigma \in \mathcal{C}$ or $\pi \ominus \sigma \in \mathcal{C}$, and

⁶For example, take $A = \{a_1, a_2, \dots\}$ to be an infinite antichain and let \mathcal{C} denote the class of permutations contained in any permutation of the form $a_{i_1} \oplus a_{i_2} \oplus \cdots$ where $1 \leq i_1 < i_2 < \cdots$. Suppose to the contrary that \mathcal{C} is \mathcal{D} -griddable for a grid irreducible subclass $\mathcal{D} \subseteq \mathcal{C}$, which then, by Proposition 3.3, cannot contain any member of A . This, however, implies that each a_i contains a basis element of \mathcal{D} , and thus \mathcal{C} contains arbitrarily long direct sums of basis elements of \mathcal{D} and so is not \mathcal{D} -griddable by Theorem 3.1.

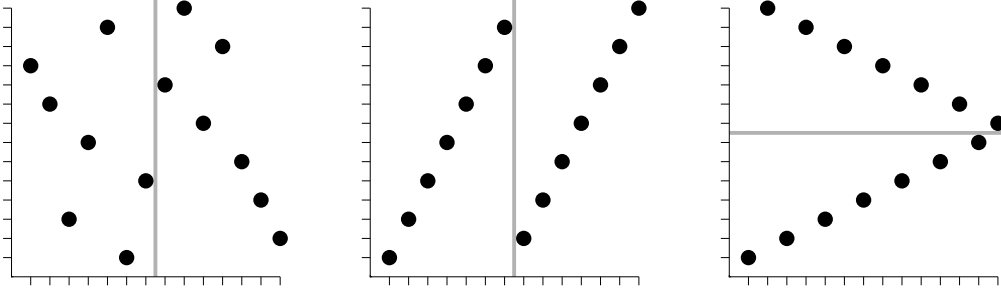


Figure 7: From left to right, an arbitrary alternation, a parallel alternation, and a wedge alternation

(3) \mathcal{C} is either sum or skew sum closed.

Proof. If a permutation class is either sum or skew sum closed then it satisfies the joint embedding property, so is atomic by Theorem 1.12, and also satisfies the conditions of Proposition 3.3, so is grid irreducible, verifying that (3) implies (1).

Suppose now that the class \mathcal{C} is both atomic and grid irreducible. For any two permutations $\pi, \sigma \in \mathcal{C}$, \mathcal{C} contains a permutation $\tau \geq \pi, \sigma$ because it is atomic. Then because \mathcal{C} is grid irreducible, it contains either $\tau \oplus \tau$ or $\tau \ominus \tau$, and thus either $\pi \oplus \sigma \in \mathcal{C}$ or $\pi \ominus \sigma \in \mathcal{C}$, so (1) implies (2).

To show that (2) implies (3) and complete the proof, list the members of \mathcal{C} as π_1, π_2, \dots (which can be done because permutation classes are countable) and for each k choose a permutation $\sigma_k \in \mathcal{C}$ which contains π_1, \dots, π_k (that σ_k exists is guaranteed by (2)). Now set $\tau_1 = \sigma_1$ and for $k \geq 1$, take $\tau_{k+1} \in \mathcal{C}$ to be either $\tau_k \oplus \sigma_{k+1}$ or $\tau_k \ominus \sigma_{k+1}$, depending on which permutation lies in \mathcal{C} . Either infinitely many of the τ_k s are formed by sums or infinitely many are formed by skew sums (or both); in the former case \mathcal{C} is sum closed, in the latter, skew sum closed. \square

4. A GRIDDING CONDITION FOR SMALL CLASSES

Our ultimate goal in this section is to prove that small permutation classes lie in particularly tractable grid classes. To motivate that work, we first review and give a new proof for a generalization of a result from [20].

Given a set of points in the plane, we define their *rectangular hull* to be the smallest axes-parallel rectangle containing them. We also define an *alternation* to be a permutation in which every odd entry lies to the left of every even entry, or any symmetry of such a permutation. Another definition is that an alternation is a permutation whose plot can be divided into two parts, by a single horizontal or vertical line, so that for every pair of points from the same part there is a point from the other part which *separates* them, i.e., there is a point from the other part which lies either horizontally or vertically between them. A *parallel alternation* is an alternation in which these two sets of entries form monotone subsequences, either both increasing or both decreasing, while a *wedge alternation* is

one in which the two sets of entries form monotone subsequences pointing in opposite directions. By the Erdős-Szekeres Theorem, if a class contains arbitrarily long alternations then it also contains arbitrarily long parallel or wedge alternations; from this observation it is easy to see, if it were not before, that if the permutation class \mathcal{C} contains arbitrarily long alternations then $\underline{\text{gr}}(\mathcal{C}) \geq 2$. (The details can also be found in Subsection A.1.) Finally, note that every parallel alternation is either simple or nearly so (i.e., the removal of one or two points gives a simple permutation), whereas wedge alternations are not simple. Long alternations are the only obstacles to an especially nice type of gridding, as shown below.

Theorem 4.1. *If the permutation class \mathcal{C} is \mathcal{D} -griddable and does not contain arbitrarily long alternations, then \mathcal{C} is \mathcal{M} -griddable for a matrix \mathcal{M} in which $G_{\mathcal{M}}$ is edgeless and every vertex is labeled by \mathcal{D} . In particular, these conditions hold whenever $\underline{\text{gr}}(\mathcal{C}) < 2$.*

Proof. Suppose that \mathcal{C} is \mathcal{N} -griddable for a $t \times u$ matrix \mathcal{N} whose entries are all subclasses of \mathcal{D} (or, without loss, equal to \mathcal{D} itself), and that \mathcal{C} does not contain any alternations of length $2m$.

It suffices to establish that there exist constants J_1 and J_2 , depending only on \mathcal{C} , so that every $\pi \in \mathcal{C}$ has an \mathcal{M} -gridding for some matrix \mathcal{M} of the desired form and of dimension at most $J_1 \times J_2$. The proof will then follow because there are only finitely many such matrices, and thus, as in the proof of Proposition 2.2, the direct sum of all of them will satisfy the conclusion of the theorem.

To this end choose an \mathcal{N} -gridded permutation $\pi \in \mathcal{C}$. We say that a rectangle R is *separated* if it is axes-parallel, $\pi(R)$ is completely contained in one cell of the chosen gridding of π , and $\pi(R)$ contains (at least) two entries which are separated by an entry in a different cell. Let \mathcal{R} denote the set of separated rectangles for the chosen gridding of π .

Suppose first that \mathcal{R} contains an independent set of size $4mtu$. Then at least $4m$ of those rectangles will lie completely within some cell, say (k, ℓ) , of the chosen gridding of π . Each of these separated rectangles contains two entries that are separated by an entry in another cell, and thus there is a subset of at least m of these rectangles that are separated by entries in the same direction (direction here meaning one of {left, right, up, down}); let us suppose that these entries are all separated by points above them, i.e., each pair is separated by an entry in a cell (k, ℓ^+) for some $\ell^+ > \ell$. However, we now have that the leftmost points of these separated rectangles, each together with a point that separated them from their partners, form an alternation of length $2m$.

Therefore, since we have assumed that \mathcal{C} has no alternations of this length, \mathcal{R} cannot contain an independent set of size $4mtu$, and thus by Lemma 3.2, \mathcal{R} can be sliced by a set, say \mathcal{L} , of at most $f(4mtu)$ vertical and horizontal lines. Consider the refined gridding of π given by the chosen \mathcal{N} -gridding together with the lines \mathcal{L} . Note that this gridding is of dimension at most $t + f(4mtu) \times u + f(4mtu)$, which we denote by $K_1 \times K_2$.

Now for each (k, ℓ) let $H_{(k, \ell)}$ denote the rectangular hull of the points that lie in the (k, ℓ) cell of this refined gridding of π , and let \mathcal{H} denote the collection of these $H_{(k, \ell)}$ hulls. Let \mathcal{S} denote the set of vertical and horizontal lines which coincide with the sides of these hulls. Note that, by their very definition, the hulls in \mathcal{H} have points on each of their sides (and

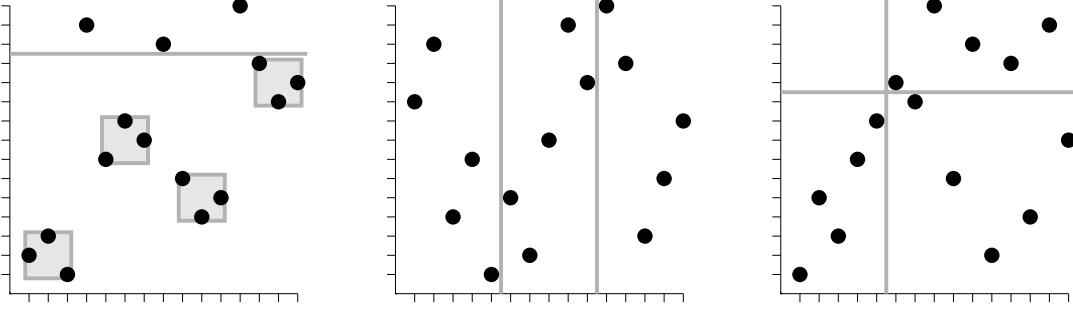


Figure 8: From left to right, a $(3, 1)$ -alternation, linear triple alternation, and hook triple alternation.

also, because π is a permutation, two disjoint hulls cannot both have points on the same vertical or horizontal line). Thus, were two of these hulls to overlap either horizontally or vertically, both would contain separated rectangles, a contradiction. In other words, \mathcal{H} is an independent set. Now let \mathcal{M} denote the matrix which corresponds to the gridding of π given by the lines \mathcal{S} , i.e., the matrix in which $\mathcal{M}_{k,\ell}$ is equal to \mathcal{D} if the (k, ℓ) cell of the refined gridding of π given by the lines \mathcal{S} is nonempty, and equal to \emptyset otherwise.

Since \mathcal{H} comes from a $K_1 \times K_2$ gridding of π , $|\mathcal{H}| \leq K_1 K_2$. Hence \mathcal{M} is of dimension at most $2K_1 K_2 + 1 \times 2K_1 K_2 + 1$. To see that \mathcal{M} satisfies the conditions of the theorem note that, because \mathcal{H} forms an independent set, the refined gridding of π contains at most one nonempty cell per row and per column. \square

The main theorem of this section, below, extends Theorem 4.1 to handle larger classes. The terms involved are defined immediately after the statement.

Theorem 4.2. *If the permutation class \mathcal{C} is \mathcal{D} -griddable and does not contain arbitrarily long $(3, 1)$ or triple alternations then \mathcal{C} is \mathcal{M} -griddable for a matrix \mathcal{M} whose graph satisfies*

- (1) *every vertex is labeled by \mathcal{D} , the class of monotone permutations, or is empty,*
- (2) *every nonisolated vertex is labeled by the class of monotone permutations, and*
- (3) *there are no connected components containing more than two vertices.*

In particular, these conditions hold whenever $\underline{\text{gr}}(\mathcal{C}) < 1 + \sqrt{2} \approx 2.41421$.

We prove Theorem 4.2 in two steps. First we show that if a class does not have a gridding satisfying condition (2) then it contains arbitrarily long $(3, 1)$ -alternations, which is our term for permutations that can be divided into two parts A and B , say, so that part A consists of nonmonotone intervals of length three, each separated from every other by at least one point in part B , and every pair of points in part B is separated by at least one of the intervals of part A (see Figure 8). In Subsection A.3 we prove that if the permutation class \mathcal{C} contains arbitrarily long $(3, 1)$ -alternations then $\underline{\text{gr}}(\mathcal{C}) \geq 1 + \sqrt{2}$.

After that, we show that if a class does not satisfy (3) then it contains arbitrarily long triple alternations. There are two types of these (which are also depicted in Figure 8):

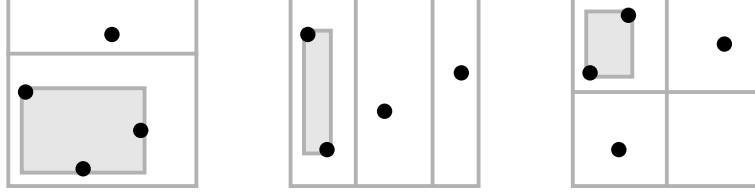


Figure 9: From left to right, a $(3, 1)$ -rectangle and two doubly separated rectangles.

- A *linear triple alternation* is one which can be divided into three parts, each with an equal number of points, so that every pair of points in one part is separated by at least one point in each of the other parts.
- A *hook triple alternation* is one that can be divided into three parts A , B , and C , again each with an equal number of points, so that no pair of points from A is separated by a point of C or vice versa, but every pair from A or C is separated by at least one point from B , and every pair from B is separated by at least one point in A and at least one point in C .

We establish in Subsection A.2 that if \mathcal{C} contains arbitrarily long triple alternations then $\underline{\text{gr}}(\mathcal{C}) \geq 1 + \varphi$ where φ denotes the golden ratio.

Proof of Theorem 4.2. The beginning of the proof mirrors that of Theorem 4.1. Suppose that \mathcal{C} is \mathcal{N} -griddable for a $t \times u$ matrix \mathcal{N} whose entries are all subclasses of \mathcal{D} and that \mathcal{C} contains neither $(3, 1)$ -alternations of length $4m$ nor triple alternations of length $3m$. Following the argument of the previous theorem, it suffices to find constants K_1 and K_2 , depending only on \mathcal{C} , so that every $\pi \in \mathcal{C}$ has an \mathcal{M} -gridding for some matrix \mathcal{M} of the desired form and of dimension at most $K_1 \times K_2$.

Choose an \mathcal{N} -gridded permutation $\pi \in \mathcal{C}$. We begin by defining a $(3, 1)$ -rectangle to be an axes-parallel rectangle R such that $\pi(R)$ is completely contained in one cell of the chosen gridding, nonmonotone, and separated by an entry from another cell of the gridding (see Figure 9). Let $\mathcal{R}_{(3,1)}$ denote the set of all $(3, 1)$ -rectangles in the chosen gridding of π .

If $\mathcal{R}_{(3,1)}$ contains an independent set of size at least $8mtu$ then, as in the proof of Theorem 4.1, at least $8m$ of those rectangles must lie completely within some cell of the gridding of π and then at least $2m$ of those rectangles must be separated in the same direction; let us suppose this direction is “up.” However, we can now find a $(3, 1)$ -alternation of length $4m$ contained in π : number the $(3, 1)$ -rectangles from left to right and take the 1st rectangle, a point separating the 2nd, the 3rd, a point separating the 4th, and so on. Therefore, as we have assumed that \mathcal{C} does not contain such permutations, $\mathcal{R}_{(3,1)}$ cannot contain an independent set of size $8mtu$, and thus $\mathcal{R}_{(3,1)}$ can be sliced by a set $\mathcal{L}_{(3,1)}$ of $f(8mtu)$ vertical and horizontal lines by Lemma 3.2.

Now we define a *doubly separated rectangle* to be an axes-parallel rectangle R such that $\pi(R)$ is completely contained in one cell of the chosen gridding of π and contains (at least) two entries which are separated by two entries, in different cells both from R and from

each other. Figure 9 shows two examples. Let \mathcal{R}_{2S} denote the set of doubly separated rectangles in π .

Suppose that \mathcal{R}_{2S} contains an independent set, \mathcal{I} , of size $mtu(t+u)^2$. At least $m(t+u)^2$ of these rectangles must lie in the same cell of the chosen gridding of π ; let \mathcal{R}'_{2S} denote a set of such doubly separated rectangles. Each doubly separated rectangle is separated by two entries in different cells, and so in \mathcal{R}'_{2S} we can find a collection, say \mathcal{R}''_{2S} , of at least m doubly separated rectangles whose separating points lie in the same two cells. However, these rectangles and separating entries give rise to a triple alternation of length at least $3m$, a contradiction.

Therefore we may assume that \mathcal{R}_{2S} does not contain an independent set of size $mtu(t+u)^2$, and thus by Lemma 3.2, it can be sliced by a collection \mathcal{L}_{2S} of at most $f(mtu(t+u)^2)$ horizontal and vertical lines.

We now return to copy the ending of the proof of Theorem 4.1. Set $\mathcal{L} = \mathcal{L}_{(3,1)} \cup \mathcal{L}_{2S}$ and consider the refined gridding of π given by the original \mathcal{N} -gridding together with these lines. This gridding is of dimension at most $t + f(8mtu) + f(mtu(t+u)^2) \times u + f(8mtu) + f(mtu(t+u)^2)$ which we again denote by $K_1 \times K_2$.

For each (k, ℓ) let $H_{(k,\ell)}$ denote the rectangular hull of the points that lie in the (k, ℓ) cell of this gridding of π , let \mathcal{H} denote the set of all these hulls, and let \mathcal{S} denote the set of vertical and horizontal lines which coincide with the sides of the hulls in \mathcal{H} . Finally, we construct a matrix \mathcal{M} corresponding to the gridding of π given by the lines \mathcal{S} by setting $\mathcal{M}_{k,\ell}$ equal to \mathcal{D} if the (k, ℓ) cell of the gridding induced by \mathcal{S} is nonmonotone, to the set of increasing permutations if it is increasing, to the set of decreasing permutations if it is decreasing, and to \emptyset if it is devoid of points.

We have that $|\mathcal{H}| \leq K_1 K_2$, so \mathcal{M} is of dimension at most $2K_1 K_2 + 1 \times 2K_1 K_2 + 1$. It remains to check this \mathcal{M} -gridding satisfies the desired conditions. By construction, no cell of the \mathcal{M} -gridding of π can contain a $(3, 1)$ - or doubly separated rectangle. Thus if $\mathcal{M}_{k,\ell}$ is nonmonotone, or equivalently, if the (k, ℓ) cell of this \mathcal{M} -gridding of π is nonmonotone, then the points in this cell (which arises from a hull in \mathcal{H}) are not separated, and thus in the graph of \mathcal{M} , the cell (k, ℓ) is isolated, as desired. If $\mathcal{M}_{k,\ell}$ is monotone, let $R_{k,\ell}$ denote the rectangular hull of the points in this cell of the \mathcal{M} -gridding of π . As we know that $R_{k,\ell}$ is not a doubly separated rectangle, it can be separated only by points in one cell. This confirms that the monotone vertices in the graph of \mathcal{M} have degree at most 1. The combination of these two assertions proves that π has a gridding of the desired form and of dimension at most $2K_1 K_2 + 1 \times 2K_1 K_2 + 1$, which therefore, by our comments at the beginning, completes the proof. \square

5. PWO AND COUNTABILITY FOR SMALL CLASSES

Theorem 4.2 says nothing about the class \mathcal{D} , and indeed, this choice is a rather delicate matter. If one chooses \mathcal{D} to be too large (e.g., the set of all permutations) then Theorem 4.2 says nothing. On the other hand, if one chooses a meager class \mathcal{D} (e.g., $\{1\}$), then there will be very few \mathcal{D} -griddable classes to which it applies. Before giving our choice for \mathcal{D} , we

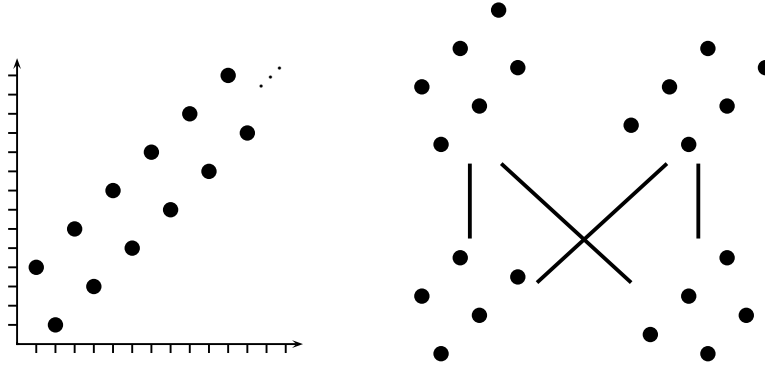


Figure 10: A plot of the increasing oscillating sequence (left) and a selection of the Hasse diagram of increasing oscillations (right).

must introduce the *increasing oscillating sequence*, which is the infinite sequence

$$4, 1, 6, 3, 8, 5, \dots, 2k + 2, 2k - 1, \dots;$$

a plot is shown in Figure 10. We further define an *increasing oscillation* to be any sum indecomposable permutation that is contained in the increasing oscillation, a *decreasing oscillation* to be the reverse of an increasing oscillation, and an *oscillation* to be any permutation that is either an increasing or a decreasing oscillation. Note that for $k \neq 3$, all oscillations of length k are simple permutation. Finally, let \mathcal{O} denote the closure of the set of oscillations and \mathcal{O}_k denote the closure of the set of oscillations of length at most k .

Note that the set of increasing oscillations forms two chains under the containment order, and that each increasing oscillation of length $k \geq 4$ contains both increasing oscillations of length $k - 1$ (see Figure 10). It is also easy to verify (e.g., using Proposition 1.2) that if π is an increasing oscillation of length $k \geq 3$ then the nonempty sum indecomposable permutations (i.e., the increasing oscillations) contained in π have the generating function

$$x + x^2 + 2x^3 + \dots + 2x^{k-1} + x^k = \frac{x + x^3 - x^k - x^{k+1}}{1 - x},$$

so $\bigoplus \pi$ has the generating function

$$\frac{1 - x}{1 - 2x - x^3 + x^k + x^{k+1}},$$

from which it follows, using Pringsheim's Theorem, that $\text{gr}(\bigoplus \pi) < \kappa$. Therefore, since we want to find a class \mathcal{D} for which every class with lower growth rate less than κ has a \mathcal{D} -gridding, \mathcal{D} must contain all increasing oscillations, and by symmetry, all decreasing oscillations, i.e., we must have $\mathcal{O} \subseteq \mathcal{D}$. This, however, is not enough: 1432 does not lie in \mathcal{O} , yet $\bigoplus 1432$ has the generating function $1/(1 - x - x^2 - x^3)$ and thus the growth rate $1.83928\dots$, much less than our goal.

We instead consider the wreath closures $\mathcal{W}(\mathcal{O}_k)$. First, however, we need to introduce a few technical results regarding wreath closures, simple permutations, and oscillations.

Proposition 5.1. *If G_π is an induced path then π is an increasing oscillation.*

Proof. Suppose that G_π is an induced path and label its vertices $\{p_1 \sim p_2 \sim \dots\}$. Note that π is an increasing oscillation if and only if its inverse is, and thus we assume by symmetry that p_2 lies below and to the right of p_1 . The third vertex, p_3 (if it exists) must then be adjacent to p_2 but not to p_1 so must lie horizontally between p_1 and p_2 and vertically above p_1 . The fourth vertex p_4 (again, if it exists) must lie vertically between p_1 and p_3 and horizontally to the right of p_2 . Continuing in this manner it is easy to see that π is contained in the increasing oscillating sequence, as desired. \square

Proposition 5.2 (Albert and Atkinson [1]). *The basis of the wreath closure of the permutation class \mathcal{C} consists of all minimal simple permutations not contained in \mathcal{C} .*

Proof. Suppose first that β is a nonsimple basis element of $\mathcal{W}(\mathcal{C})$ and express β as the inflation $\sigma[\alpha_1, \dots, \alpha_m]$ where σ is simple. If σ or one of the α_i s is not contained in $\mathcal{W}(\mathcal{C})$ then it contains a basis element of $\mathcal{W}(\mathcal{C})$, so β is not a basis element of $\mathcal{W}(\mathcal{C})$, a contradiction. Thus β lies in $\mathcal{W}(\mathcal{C})$, again a contradiction. Therefore we conclude that the basis elements of $\mathcal{W}(\mathcal{C})$ are all simple, and thus they are the minimal simple permutations not contained in $\mathcal{W}(\mathcal{C})$ by the definition of basis, and the proposition follows by observing that $\mathcal{W}(\mathcal{C})$ and \mathcal{C} contain the same set of simple permutations. \square

While we state only the permutation case of the following result (a proof of this case is also given by Murphy [24]), Schmerl and Trotter's proof includes all irreflexive binary relational structures.

Theorem 5.3 (Schmerl and Trotter [27]). *Every simple permutation π of length $n \geq 2$ contains a simple permutation of length $n - 1$ or $n - 2$. Furthermore, π contains a simple permutation of length $n - 1$ unless π is a simple parallel alternation (recall Figure 7).*

The basis of the class of permutations contained in any increasing oscillation was stated without proof in Murphy's thesis [24], but the first proof appeared later:

Proposition 5.4 (Brignall, Ruškuc, and Vatter [12]). *The class of all permutations contained in some increasing oscillation is $\text{Av}(321, 2341, 3412, 4123)$.*

The proof of Proposition 5.4 in [12] uses an encoding of permutations known as the "rank encoding"⁷. The following result follows immediately from that work.

Proposition 5.5. *The generating function for the class of all permutations contained in some increasing oscillation is $(1 - x)/(1 - 2x - x^3)$, and thus its growth rate is κ .*

We are more interested in the basis of $\mathcal{W}(\mathcal{O})$, computed below.

Proposition 5.6. *The basis of $\mathcal{W}(\mathcal{O})$ consists of 25314, 41352, 246153, 251364, 314625, 351624, 415263, and every symmetry of one of these permutations.*

⁷We refer the reader to Albert, Atkinson, and Ruškuc [2] for a detailed study of the rank encoding.

Proof. First note that since \mathcal{O} is closed under all eight permutation class symmetries, $\mathcal{W}(\mathcal{O})$ — and thus the basis of $\mathcal{W}(\mathcal{O})$ — must be as well. Let B denote the basis specified in the statement of the theorem. It can be checked that each of the 7 permutations listed are basis elements of $\mathcal{W}(\mathcal{O})$, so it suffices to prove that $\mathcal{W}(\mathcal{O})$ has no additional basis elements. By Proposition 5.2, this amounts to proving that every simple permutation not contained in \mathcal{O} contains an element of B . Let π denote a simple permutation not contained in \mathcal{C} . We prove the fact by induction on the length, n , of π ; as it is easy to check for $n \leq 6$, we will assume that $n \geq 7$.

If π is a simple parallel alternation then π contains at least one of 246135, 362514, 415263, or 531642, which are all symmetries of 415263, so π contains an element of B , as desired.

Thus we may assume that π is not a parallel alternation, and thus by Theorem 5.3, π contains a simple permutation of length $n - 1$. Label the indices of this simple permutation $1 \leq i_1 < \dots < i_{n-1} \leq n$ and let $i = [n] \setminus \{i_1, \dots, i_{n-1}\}$ denote the index of the missing entry. By the minimality of π , these entries must be order isomorphic to an oscillation, and by symmetry we may assume that they are order isomorphic to $4, 1, 6, 3, \dots, 2k + 2, 2k - 1$ if $n - 1$ is even or to $4, 1, 6, 3, \dots, 2k, 2k - 3, 2k - 1$ if $n - 1$ is odd.

It is clear that if $i < i_1$ or $i > i_3$ then the permutation obtained from π by removing the entry in position i_2 must still be simple and not order isomorphic to an oscillation; thus it must contain an element of B by induction, so π does as well. This also occurs if $\pi(i) > \pi(i_5)$, so we may assume that $i_1 < i < i_3$ and $\pi(i) < \pi(i_5)$. In this case, it is clear that the permutation obtained from π by removing the entry in position i_{n-2} if n is even and i_{n-1} if n is odd is simple and not order isomorphic to an oscillation, and thus we are done again by induction, completing the proof. \square

As a consequence of this basis result, we see that all small classes have $\mathcal{W}(\mathcal{O})$ griddings.

Proposition 5.7. *Every class with lower growth rate at most the unique positive real root of $1 + 3x + 3x^2 + 2x^3 + x^4 + x^5 - x^6$, ≈ 2.24409 , has a $\mathcal{W}(\mathcal{O})$ -gridding.*

Proof. Theorem 3.1 shows that the permutation class \mathcal{C} fails to have a $\mathcal{W}(\mathcal{O})$ -gridding if and only if it contains either $\oplus\beta$ or $\ominus\beta$ for some basis element β of $\mathcal{W}(\mathcal{O})$. Therefore we need only compute (via Proposition 1.3) the growth rates of $\oplus\beta$ and $\ominus\beta$ for each basis element β specified in the preceding proposition. The smallest such classes are the sum completion of 251364 and the skew sum completion of its reverse. The generating function for the nonempty sum indecomposable permutations contained in 251364 is $x + x^2 + 2x^3 + 3x^4 + 3x^5 + x^6$ (not counting the empty permutation), and thus the generating function for the sum completion is $1/(1 - x - x^2 - 2x^3 - 3x^4 - 3x^5 - x^6)$, which has the growth rate stated in the proposition. \square

The following two propositions then follow routinely.

Proposition 5.8. *Every permutation class with lower growth rate less than κ is $\mathcal{W}(\mathcal{O}_k)$ -griddable for some k .*

Proof. If \mathcal{C} is a permutation class with growth rate less than κ then Proposition 5.7 shows that \mathcal{C} is $\mathcal{W}(\mathcal{O})$ -griddable. If \mathcal{C} were to contain every increasing oscillation or every decreasing oscillation, then Proposition 5.5 would imply that $\underline{\text{gr}}(\mathcal{C}) \geq \kappa$, a contradiction. Thus there must be some k for which \mathcal{C} has a $\mathcal{W}(\mathcal{O}_k)$ gridding. \square

Proposition 5.9. *Let \mathcal{D} be a pwo permutation class and suppose that the matrix \mathcal{M} satisfies the conclusion of Theorem 4.2 with respect to \mathcal{D} . Then $\text{Grid}(\mathcal{M})$ (and thus each of its subclasses) is also pwo. In particular, by Theorem 4.2 and Proposition 5.8, every permutation class with lower growth rate less than κ is pwo.*

Proof. By Proposition 2.6 it suffices to show that every connected component of $\text{Grid}(\mathcal{M})$ is pwo. By assumption, these are either \mathcal{D} or edges in which both cells are labeled by monotone classes. The proof is then completed by noting that both classes are pwo (by assumption and Theorem 2.5, respectively). \square

We are now ready to answer Question 2.

Theorem 5.10. *There are only countably many permutation classes with lower growth rate less than κ .*

Proof. We have already noted that each of these classes lies in $\text{Grid}(\mathcal{M})$ for some matrix \mathcal{M} which satisfies the conclusion of Theorem 4.2 with respect to $\mathcal{W}(\mathcal{O}_k)$ for some k . Note that for every k there are only countably many matrices of this form. Now consider a fixed matrix \mathcal{M} of this form and let $\mathfrak{I}_{\mathcal{M}}$ denote the set of permutation classes contained in $\text{Grid}(\mathcal{M})$. Proposition 5.9 shows that $\text{Grid}(\mathcal{M})$ is pwo, so $\mathfrak{I}_{\mathcal{M}}$ is countable by Proposition 1.8. Thus we have that the set of permutation classes with lower growth rate less than κ is the countable union of a countable union of countable sets $\bigcup_{k \geq 1} \bigcup_{\mathcal{M}} \mathfrak{I}_{\mathcal{M}}$ where the inner union is over all matrices \mathcal{M} satisfying Theorem 4.2 with respect to $\mathcal{W}(\mathcal{O}_k)$, proving the theorem. \square

6. THE GROWTH RATES BELOW κ

As we now have, by Proposition 5.9, that small classes are pwo, we can apply Propositions 1.7, 1.14, and 3.4 to study their possible growth rates.

Theorem 6.1. *If the permutation class \mathcal{C} satisfies $\underline{\text{gr}}(\mathcal{C}) < \kappa$ then $\text{gr}(\mathcal{C})$ exists and is equal to $\text{gr}(\mathcal{C}')$ for a subclass $\mathcal{C}' \subseteq \mathcal{C}$ that is either sum closed, skew sum closed, or contained in a 1×2 or 2×1 monotone grid class.*

Proof. First set $\mathcal{C}^0 = \mathcal{C}$ and choose an atomic subclass $\mathcal{A}^0 \subseteq \mathcal{C}^0$ for which $\overline{\text{gr}}(\mathcal{A}^0) = \overline{\text{gr}}(\mathcal{C}^0)$; such a class exists by Proposition 1.14 because \mathcal{C}^0 is pwo. As \mathcal{A}^0 is itself pwo, Proposition 3.4 shows that \mathcal{A}^0 is \mathcal{D}^0 -griddable for a grid irreducible class $\mathcal{D}^0 \subseteq \mathcal{A}^0$. To complete the base case, note that $\underline{\text{gr}}(\mathcal{A}^0) \leq \underline{\text{gr}}(\mathcal{C}^0) < 1 + \sqrt{2}$ and so \mathcal{A}^0 is \mathcal{M}^0 -griddable for a matrix \mathcal{M}^0 which satisfies the conditions of Theorem 4.2 with respect to \mathcal{D}^0 .

For each $i \geq 1$ we begin by letting \mathcal{C}^i denote a restriction of \mathcal{A}^{i-1} to a connected component of \mathcal{M}^{i-1} which satisfies $\overline{\text{gr}}(\mathcal{C}^i) = \overline{\text{gr}}(\mathcal{A}^{i-1})$, noting that such a class exists by Proposition 2.7. If this connected component is of size two then, by our conditions on \mathcal{M}^{i-1} , \mathcal{C}^i is contained in a 1×2 or 2×1 monotone grid class and we are done.

Let us therefore suppose this connected component consists of a single cell of \mathcal{M}^{i-1} and thus $\mathcal{C}^i \subseteq \mathcal{D}^{i-1}$. Now choose an atomic class $\mathcal{A}^i \subseteq \mathcal{C}^i$ with $\overline{\text{gr}}(\mathcal{A}^i) = \overline{\text{gr}}(\mathcal{C}^i)$ and a grid irreducible subclass $\mathcal{D}^i \subseteq \mathcal{A}^i$ so that \mathcal{A}^i is \mathcal{D}^i -griddable. Finally, choose a matrix \mathcal{M}^i satisfying the conditions of Theorem 4.2 with respect to \mathcal{D}^i for which \mathcal{A}^i is \mathcal{M}^i -griddable.

In this process we create a descending chain of classes

$$\mathcal{C} = \mathcal{C}^0 \supseteq \mathcal{A}^0 \supseteq \mathcal{D}^0 \supseteq \mathcal{C}^1 \supseteq \mathcal{A}^1 \supseteq \mathcal{D}^1 \supseteq \cdots,$$

all with identical upper growth rates. As \mathcal{C} is pwo this chain must terminate by Proposition 1.7; let h denote the least integer such that $\mathcal{C}^{h+1} = \mathcal{C}^h$.

Setting $\mathcal{C}' = \mathcal{C}^h = \mathcal{A}^h = \mathcal{D}^h$ we see that \mathcal{C}' is both grid irreducible and atomic and thus, by Proposition 3.5, either sum or skew closed, and by construction, $\overline{\text{gr}}(\mathcal{C}) = \overline{\text{gr}}(\mathcal{C}')$. Proposition 1.4 shows that $\text{gr}(\mathcal{C}')$ exists, and thus we must have $\underline{\text{gr}}(\mathcal{C}) = \overline{\text{gr}}(\mathcal{C}) = \text{gr}(\mathcal{C}')$, so $\text{gr}(\mathcal{C})$ exists and is equal to $\text{gr}(\mathcal{C}')$, completing the proof. \square

Theorem 6.1 reduces our search for growth rates below κ to small monotone grid classes and (by symmetry) sum closed classes. The monotone grid class case is handled by Proposition A.1, which shows that such classes have a growth rate of 0, 1, or 2. The sum closed classes are also, provided they are not too complicated, easily enumerated using Proposition 1.3. First, however, we need to investigate sum indecomposable permutations themselves. We begin by strengthening Proposition 1.2.

Proposition 6.2. *The permutation π of length n is sum indecomposable if and only if G_π contains a path connecting the vertices 1 and n .*

Proof. If π is sum indecomposable then G_π is connected by Proposition 1.2, so that direction follows easily. Now suppose that G_π contains a path connecting the vertices 1 and n . Let $P = \{1 = p_1 \sim p_2 \sim \cdots \sim p_m = n\}$ denote a shortest path between the vertices 1 and n , so P is an induced path. By Proposition 5.1, the points p_1, p_2, \dots, p_m are order isomorphic to an increasing oscillation.

We then have that every entry above and to the left of a $p_{2\ell}$ entry is adjacent to $p_{2\ell}$ in G_π while every entry below and to the right of a $p_{2\ell-1}$ is adjacent to $p_{2\ell-1}$ in G_π . It is easy to check that at least one of these conditions holds for each entry of π , establishing that G_π is connected, as desired. \square

Finally we have reached the fact we need.

Proposition 6.3. *For a sum indecomposable permutation π of length n either*

- (1) $\pi = n \cdots 21, 12 \cdots (n-1) \ominus 1$, or $1 \ominus 12 \cdots (n-1)$, or
- (2) π contains two distinct sum indecomposable permutations of length $n-1$.

In particular, if the permutation class \mathcal{C} contains 2 sum indecomposable permutations of length $n \geq 4$ then it also contains 2 sum indecomposable permutations of length $n - 1$.

Proof. Take π to be a sum indecomposable permutation of length n and denote the indices of the lexicographically minimal path connecting 1 to n in G_π by $1 = i_1 < i_2 < \dots < i_m = n$. If this path contains only the vertices 1 and n then $\pi(1) > \pi(n)$ and it is easy to see that (2) is satisfied if any of the following hold:

- π contains entries both above $\pi(1)$ and below $\pi(n)$,
- the entries above $\pi(1)$ are nonmonotone,
- the entries below $\pi(n)$ are nonmonotone, or
- the entries lying vertically between $\pi(1)$ and $\pi(n)$ are nonmonotone.

If none of these conditions hold, it follows either that π satisfies (1) or that $\pi = 1 \ominus 12 \dots (n-2) \ominus 1$, which satisfies (2).

Thus we may suppose that $m \geq 3$. Now we divide π into the sections $\pi((i_k, i_{k+1}) \times [n])$. If all of these sections are empty then G_π is a path, so π is an increasing oscillation by Proposition 5.1, and it is easily checked that (2) holds. Similarly, (2) is clearly satisfied if two of these sections are nonempty. Thus we may assume that precisely one of these sections, say $\pi((i_j, i_{j+1}) \times [n])$, is nonempty. In this (final) case, (2) can be seen to hold by the following argument: removing an entry from $\pi((i_j, i_{j+1}) \times [n])$ gives one sum indecomposable permutation, and removing either the leftmost, rightmost, top, or bottom entry gives the other. \square

Theorem 6.4. *The sub- κ growth rates of permutation classes consist precisely of 0 and the following families:*

	sequence of sum indecomposables	growth rate = the largest root of
(I)	$(1 \times k)$	$1 - 2x^k + x^{k+1}$
(II)	$(1 \times \infty)$	$2 - x$
(III)	$(1, 1, 2 \times k, 3)$	$3 - x - x^{k+1} + x^{k+4} - 2x^{k+3}$
(IV)	$(1, 1, 2 \times k, 3, 1)$	$1 + 2x - x^2 - x^{k+2} + x^{k+5} - 2x^{k+4}$
(V)	$(1, 1, 2 \times k, 1 \times \ell)$	$1 + x^\ell - x^{k+\ell} - 2x^{k+\ell+2} + x^{k+\ell+3}$
(VI)	$(1, 1, 2 \times k, 1 \times \infty)$	$1 - x^k - 2x^{k+2} + x^{k+3}$

The smallest of these that is greater than 2, $\nu \approx 2.06599$, occurs when $(k, \ell) = (1, 1)$ in family (V). Furthermore, the growth rates of type (V) accumulate at growth rates of type (VI) which themselves accumulate at κ , so κ is the smallest accumulation point of accumulation points of growth rates of permutation classes.

Proof. By Theorem 6.1 it suffices to consider sum closed classes, so suppose that \mathcal{C} is a sum closed class with $\text{gr}(\mathcal{C}) < \kappa$. Let s_n denote the number of sum indecomposable permutations of length $n \geq 1$ contained in \mathcal{C} , so the generating function for \mathcal{C} is $1/(1 - \sum s_n x^n)$ by

Proposition 1.3. Let us say that the sequence (s_n) is *large* if it leads to a growth rate at least κ and *small* otherwise. We also write $(s'_n) \geq (s_n)$ if $s'_n \geq s_n$ for all $n \geq 1$; note that if (s_n) is large and $(s'_n) \geq (s_n)$ then (s'_n) must also be large.

We begin by establishing several large sequences. As the calculations are similar in each case, we will give the details in only one, showing that the sequence

$$(1, 1, \underbrace{2, \dots, 2}_k, 4),$$

which we abbreviate as $(1, 1, 2 \times k, 4)$, is large for all $k \geq 0$ (although, when $k = 0$ this sequence is impossible to achieve because there are only 3 sum indecomposable permutations of length 3). The generating function for the corresponding class is, by Proposition 1.3,

$$f = \frac{1}{1 - x - x^2 - 2x^3 - \dots - 2x^{k+2} - 4x^{k+3}} = \frac{1 - x}{1 - 2x - x^3 - 2x^{k+3} + 4x^{k+4}}.$$

By Pringsheim's Theorem, the growth rate of this class is $1/R$ where R denotes the smallest positive root of the denominator of f . Equivalently, setting $z = 1/x$ and multiplying by z^{k+4} , the growth rate of this class is equal to the largest positive root of

$$\begin{aligned} g &= z^{k+4} \left(1 - 2(1/z) - (1/z)^3 - 2(1/z)^{k+3} + 4(1/z)^{k+4} \right) \\ &= z^{k+4} - 2z^{k+3} - z^{k+1} - 2z + 4 \\ &= 4 - 2z - z^{k+1} (1 + 2z^2 - z^3). \end{aligned}$$

Since $g(\kappa) = 4 - 2\kappa < 0$ and the leading coefficient of g is positive, g has a real root greater than κ , establishing that the sequence $(1, 1, 2 \times k, 4)$ is large. Therefore if any sequence (s_n) has an entry $s_{k+3} \geq 4$ for any k then Proposition 6.3 implies that $(s_n) \geq (1, 1, 2 \times k, 4)$, so (s_n) is large by the computation above. Similar computations show that the sequences $(1, 1, 2 \times k, 3, 1, 1)$, $(1, 1, 2 \times k, 3, 2)$, and $(1, 1, 2 \times \infty)$ are large for all $k \geq 0$. This leaves only the few possibilities for small sequences listed in the statement of the theorem. \square

7. CONCLUDING REMARKS

Other objects. Growth rates of hereditary properties of other types of combinatorial structure (i.e., permutation classes in our context) have received considerable attention recently. The case of ordered graphs, studied by Balogh, Bollobás, and Morris [9], seems particularly similar to the permutation case. Note that while the sub-2 growth rates of permutation classes and of hereditary properties of ordered graphs are identical, this is not the case above 2; there is a hereditary property of ordered graphs whose growth rate is the largest real root of $1 + 2x + x^2 + x^3 + x^4 - x^5$, ≈ 2.03166 , while we have proved that this is not a growth rate of any permutation class. (Conversely, every growth rate of a permutation

class is the growth rate of a hereditary property of ordered graphs, simply consider ordered versions of the graphs G_π for permutations π in the class.)

Growth rates beyond κ . Albert and Linton [3] have recently shown that there is a perfect uncountable set of growth rates of permutation classes between the unique positive root of $3 + 4x + 2x^2 + 2x^3 + x^4 - x^5$, ≈ 2.47665 , and 3, thereby disputing several conjectures of Balogh, Bollobás, and Morris [9].

It remains to be seen if there is real number γ such that the set of growth rates of permutation classes contains all real numbers at least γ .

Extensions. Many of our techniques could, with effort, be extended to deal with larger permutation classes. In particular, Theorem 4.2, which restricts the possible gridding matrices of small permutation classes, seems ripe for extension. Also, surely the $\mathcal{W}(\mathcal{O}_k)$ classes we use could be replaced, and the analysis involved in Theorem 6.4 could be extended as far as one's patience, although this would also require extending Proposition 6.3.

The major obstacle to extension is Theorem 6.1. Recall that the proof of this theorem relies on finding atomic, grid irreducible subclasses of a particular type inside the class of interest. The first difficulty with this approach is that if the class we begin with is not pwo then this process need not terminate. Even if that problem were resolved, one would have to deal with the fact that Propositions 1.14 and 3.4 hold only for pwo classes. For these reasons, it seems that our techniques cannot extend beyond κ unless arbitrary permutation classes can be approximated by pwo permutation classes.

This problem, approximating classes by pwo classes, is essentially an interchange of limits problem; one could imagine that given any permutation class \mathcal{C} , there is some chain $\mathcal{C}^1 \subseteq \mathcal{C}^2 \subseteq \dots$ of pwo classes, all contained in \mathcal{C} and satisfying $\mathcal{C}^i \rightarrow \mathcal{C}$, such that $\overline{\text{gr}}(\mathcal{C}^i) \rightarrow \overline{\text{gr}}(\mathcal{C})$, and similarly for $\underline{\text{gr}}$, i.e., that for these (presumably) carefully chosen subclasses,

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n^i|} = \limsup_{n \rightarrow \infty} \lim_{i \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n^i|}.$$

Furthermore, this author feels strongly that the set of (upper, lower, proper) growth rates of pwo classes ought to be closed under limits, and so offers:

Conjecture 7.1. *Every (upper, lower, proper) growth rate of a permutation class is achieved by a pwo permutation class.*

Another natural question is the following.

Question 7.2. *Is there a (upper, lower, proper) growth rate of a permutation class that is not achieved by a sum closed permutation class?*

The existence of growth rates. Of particular irritation is the fact that arbitrary permutation classes are not known to have growth rates. Our work has modestly improved the situation (now we know that sub- κ classes have growth rates, while before the bound was

2), but it seems possible that extensions to it could manage more. If Theorem 4.2 were extended arbitrarily far, then, albeit with significant adaptation, Theorem 6.1 could be extended to show that all pwo permutation classes have growth rates. Extending that result to arbitrarily classes might then be possible, depending on if and how nicely such classes can be approximated by pwo classes.

The decision problem. Our proof shows that it is decidable whether a permutation class \mathcal{C} given by its finite basis satisfies $\text{gr}(\mathcal{C}) < \kappa$. All one needs to check is whether \mathcal{C} is $\mathcal{W}(\mathcal{O})$ -griddable, which can be done using Proposition 5.6 and Theorem 3.1, and whether \mathcal{C} contains arbitrarily long increasing or decreasing oscillations, or $(3, 1)$ or triple alternations, which is also tractable.

Finite bases for small classes. We doubt, but have been unable to rule out, the existence of infinitely based permutation classes of growth rate less than κ . This too is very much an interchange of limits problem. Suppose that there were an infinitely based class, say $\text{Av}(\beta_1, \beta_2, \dots)$, with growth rate less than κ . Because bases are antichains and we have proved, in Proposition 5.9, that classes with lower growth rate less than κ are pwo, we would have to have $\underline{\text{gr}}(\text{Av}(\beta_1, \dots, \beta_m)) > \kappa$ for all m , which seems unlikely.

Enumeration below κ . Thus far we have ignored the exact enumeration problem, not out of neglect but rather complete ignorance. Permutation classes of growth rate 1 are known to have eventually polynomial enumeration (first established by Kaiser and Klazar [21], this result follows rather quickly from the results presented here, particularly Theorem 4.1 applied to the monotone griddings that such classes must have; see [20] for the missing details).

Moving beyond the very small classes, Albert, Linton, and Ruškuc [4] have introduced a correspondence between permutation classes and formal languages, known as the insertion encoding, in which every class which does not contain arbitrarily long alternations corresponds to a regular language⁸. As no class with a growth rate below 2 can contain arbitrarily long alternations, these all have rational generating functions. It is thus natural to ask for the smallest class with an *irrational* generating function.

Conjecture 7.3. *Every permutation class with growth rate less than κ has a rational generating function.*

Note that there are classes with growth rate at most κ which have not only irrational, but nonholonomic, generating functions⁹. However, our belief in Conjecture 7.3 is based

⁸This is only a special case of the applicability of the insertion encoding.

⁹**Proposition 7.4** (Atkinson and Stitt [8]; Murphy [24, Chapter 9]). *If the permutation class \mathcal{C} contains an infinite antichain then it also contains a subclass with a nonholonomic generating function.*

Proof. Choose an infinite antichain $A \subseteq \mathcal{C}$ that has at most one element of each length. If $A_1 \neq A_2$ are two subsets of A then the two subclasses $\mathcal{C} \cap \text{Av}(A_1)$ and $\mathcal{C} \cap \text{Av}(A_2)$ have different enumerations. Because A is infinite, this gives 2^{\aleph_0} different generating functions. Now notice that if f is a holonomic generating

on more than coincidence. Suppose that \mathcal{C} is a permutation class with $\text{gr}(\mathcal{C}) < \kappa$, so \mathcal{C} is \mathcal{M} -griddable for a matrix \mathcal{M} satisfying the conditions of Theorem 4.2. Then every restriction of \mathcal{C} to a connected component of \mathcal{M} can be enumerated by the insertion encoding¹⁰. The only stumbling block is that permutations often have multiple \mathcal{M} -griddings, but they cannot have *too* many, and this is exactly the sort of problem one expects to be able to handle with regular languages.

Axes-parallel rectangles. Lemma 3.2 provides an exponential (in terms of the independence number) bound on the number of lines needed to slice a collection of axes-parallel rectangles, but surely this bound is not tight. In particular, this author does not know of an example demonstrating that a linear bound will not suffice.

A. CALCULATIONS

In the calculations that follow we make frequent use of regular languages, and the reader is referred to Hopcroft, Motwani, and Ullman [18] for all concepts and notation not defined here. Given a finite alphabet A , we denote by A^* the *free monoid over A* as the regular language consisting of all (possibly empty) words over A , i.e., $A^* = \{w_1 \cdots w_n : w_i \in A \text{ for all } i\}$. We further define A^+ as the language of all nonempty words over A . Given two regular languages A and B we denote their union by $A \cup B$ and their concatenation as AB (i.e., the set of all words ab for $a \in A$ and $b \in B$).

A.1. SUBCLASSES OF 1×2 AND 2×1 MONOTONE GRID CLASSES

Up to symmetry, the 1×2 and 2×1 monotone grid classes are

$$\text{Grid} \left(\begin{smallmatrix} \text{Av}(21) & \text{Av}(21) \end{smallmatrix} \right) \text{ and } \text{Grid} \left(\begin{smallmatrix} \text{Av}(21) & \text{Av}(12) \end{smallmatrix} \right).$$

Proposition 2.1 shows that we need consider only gridded permutations in order to characterize the possible growth rates of subclasses of these. As there is an order-preserving bijection between the gridded permutations in the two classes above, it suffices to consider subclasses of the closure of the parallel alternations.

An encoding studied by Atkinson, Murphy, and Ruškuc [6] and Albert, Atkinson, and Ruškuc [2] is useful for this purpose. This encoding associates to each gridded permutation π of length n the word $w_\pi = w_\pi(1) \cdots w_\pi(n)$ where $w_\pi(i) = \ell$ if the entry $\pi(i)$ lies on the left and $w_\pi(i) = r$ if the i th entry lies on the right, yielding the language $\{\ell, r\}^+$.

Suppose now that \mathcal{C} is a proper subclass of $\text{Grid} \left(\begin{smallmatrix} \text{Av}(21) & \text{Av}(21) \end{smallmatrix} \right)$, so there is some permutation β of length k contained in this class but not in \mathcal{C} . Choose some gridding of

function for a permutation class then the recurrence satisfied by f may be chosen to have integral coefficients and integral initial conditions, and so there are only countably many holonomic generating functions for permutation classes. \square

¹⁰Subclasses of $\mathcal{W}(\mathcal{O})$ do not contain arbitrarily long alternations, and the subclasses of connected components of size two are also covered by their theorem, when stated in its full generality.

β and denote the word corresponding to this gridding by w_β . For a letter $a \in \{\ell, r\}$ we denote by \bar{a} the sole element of $\{\ell, r\} \setminus \{a\}$. It follows that the set of gridded permutations in \mathcal{C} corresponds to a subset of the language

$$\overline{w_\beta(1)}^* \cup \overline{w_\beta(1)}^* w_\beta(1) \overline{w_\beta(2)}^* \cup \cdots \cup \overline{w_\beta(1)}^* w_\beta(1) \overline{w_\beta(2)}^* w_\beta(2) \cdots w_\beta(k-1) \overline{w_\beta(k)}^*,$$

which has the generating function

$$\frac{1}{1-x} + x \left(\frac{1}{1-x} \right)^2 + \cdots + x^{k-1} \left(\frac{1}{1-x} \right)^k$$

and thus the growth rate 1. Of course, since we have considered just one gridding of one basis element of \mathcal{C} , this means that the upper growth rate of \mathcal{C} can be at most 1. However, if \mathcal{C} contains either $12 \cdots n$ or $n \cdots 21$ for all n then $\underline{\text{gr}}(\mathcal{C}) \geq 1$ whereas otherwise, by the Erdős-Szekeres Theorem, $\text{gr}(\mathcal{C}) = 0$. Thus we obtain the following result.

Proposition A.1. *Every subclass of a 1×2 or 2×1 monotone grid class has growth rate 0, 1, or 2.*

A.2. TRIPLE ALTERNATIONS

We consider first the linear triple alternations, and (without loss) those which can be divided by vertical lines into three parts, left, middle, and right. Consider such a permutation π_1 , in which each of the parts have $3m^8$ entries. By the Erdős-Szekeres Theorem, the entries of the left part contain a monotone subsequence of length at least m^4 . Consider then the linear triple alternation $\pi_2 \leq \pi_1$ which contains each of these m^4 entries together with another m^4 entries from each of the other parts. By applying the Erdős-Szekeres Theorem again to the entries in the middle part, we find a monotone subsequence with m^2 entries. Now consider the linear triple alternation $\pi_3 \leq \pi_2$ which contains these m^2 entries, together with m^2 entries from each of the other two parts. By applying the Erdős-Szekeres Theorem a third time, to the entries in the right part, find a linear triple alternation with $3m$ entries in which each of the parts is monotone, giving the following result.

Proposition A.2. *If a permutation class contains arbitrarily long linear triple alternations, then it contains a symmetry of one of the following classes:*

$$\begin{aligned} &\text{Grid} \left(\begin{array}{ccc} \text{Av}(21) & \text{Av}(21) & \text{Av}(21) \end{array} \right), \\ &\text{Grid} \left(\begin{array}{ccc} \text{Av}(21) & \text{Av}(21) & \text{Av}(12) \end{array} \right), \\ &\text{Grid} \left(\begin{array}{ccc} \text{Av}(21) & \text{Av}(12) & \text{Av}(21) \end{array} \right), \text{ or} \\ &\text{Grid} \left(\begin{array}{ccc} \text{Av}(21) & \text{Av}(12) & \text{Av}(12) \end{array} \right). \end{aligned}$$

We are only interested in the lower growth rates of these classes, which by Proposition 2.1, are equal to the lower growth rates of the gridded permutations in them. Again we have that there is an order-preserving bijection between the gridded permutations in them, so it does not matter which we consider. Using the analogue of the encoding used for parallel alternations in the previous subsection, we find that their lower growth rates are all equal to 3, and thus:

Proposition A.3. *If the permutation class \mathcal{C} contains arbitrarily long linear triple alternations then $\underline{\text{gr}}(\mathcal{C}) \geq 3$.*

Now consider the hook triple alternations. The following proposition follows from the same multiple applications of Erdős-Szekeres as Proposition A.2.

Proposition A.4. *If a permutation class contains arbitrarily long hook triple alternations, then it contains a symmetry of one of the following classes:*

$$\begin{aligned} & \text{Grid} \left(\begin{array}{cc} \text{Av}(21) & \\ \text{Av}(21) & \text{Av}(21) \end{array} \right), \quad \text{Grid} \left(\begin{array}{cc} \text{Av}(21) & \\ \text{Av}(21) & \text{Av}(12) \end{array} \right), \quad \text{Grid} \left(\begin{array}{cc} \text{Av}(12) & \\ \text{Av}(21) & \text{Av}(12) \end{array} \right), \\ & \text{Grid} \left(\begin{array}{cc} \text{Av}(21) & \\ \text{Av}(12) & \text{Av}(21) \end{array} \right), \quad \text{Grid} \left(\begin{array}{cc} \text{Av}(21) & \\ \text{Av}(12) & \text{Av}(12) \end{array} \right), \text{ or } \text{Grid} \left(\begin{array}{cc} \text{Av}(12) & \\ \text{Av}(12) & \text{Av}(12) \end{array} \right). \end{aligned}$$

Again it suffices to enumerate gridded permutations in these classes, and again, the answer is the same for all of the classes, so we study

$$\text{Grid} \left(\begin{array}{cc} \text{Av}(21) & \\ \text{Av}(21) & \text{Av}(21) \end{array} \right),$$

using a special case of an encoding from Vatter and Waton [29].

We divide the entries of such a gridded hook triple alternation, say π , into those which are in the top (designated by t), right (r), and hook (h) parts. We first read the h and t entries from left-to-right, recording h s and t s, to form a word w_π^{ht} . We then read the h and r entries from bottom-to-top, recording a word w_π^{hr} . Note that, because the hook entries are monotone increasing, they correspond to the same entries in each of these two words. We use this to amalgamate the words, identifying them along their h entries; for example, suppose that

$$\begin{aligned} w_\pi^{ht} &= t^{i_1} h t^{i_2} \dots t_{i_k} h t^{i_{k+1}} \text{ and} \\ w_\pi^{hr} &= r^{j_1} h r^{j_2} \dots r_{i_k} h r^{j_{k+1}}. \end{aligned}$$

In this case our amalgamated word is

$$w_\pi = t^{i_1} r^{j_1} h t^{i_2} r^{j_2} \dots t^{i_k} r^{j_k} h t^{i_{k+1}} r^{j_{k+1}}.$$

This establishes a bijection between gridded permutations in this grid class and the language

$$\{h, r, t\}^* \setminus \{h, r, t\}^* r t \{h, r, t\}^*,$$

from which it follows that the generating function for such gridded permutations is $3x/(1 - 3x + x^2)$, yielding the following result.

Proposition A.5. *If the permutation class \mathcal{C} contains arbitrarily long hook triple alternations then $\underline{\text{gr}}(\mathcal{C}) \geq 1 + \varphi$, where φ denotes the golden ratio, approximately 1.61803.*

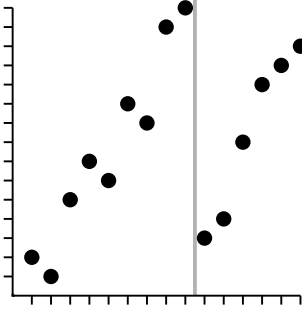


Figure 11: A gridded permutation in $\text{Grid}(\oplus 21 \text{ Av}(21))$

A.3. $(3, 1)$ -ALTERNATIONS

In this subsection our aim is to give a lower bound on the growth rate of a class which contains arbitrarily long $(3, 1)$ -alternations. Recall that such an alternation may be divided by a single horizontal or vertical line into two parts, A and B , so that A consists of nonmonotone intervals of length three, each separated from every other by at least one point in part B , and every pair of points in part B is separated by at least one of the intervals of part A . Consider $(3, 1)$ -alternation, π , of length at least $4m^4$. By symmetry we may suppose that it can be separated by a vertical line into two such parts A and B . By the Erdős-Szekeres Theorem, we then have there is a monotone subsequence of the intervals in part A containing at least m^2 of them; let us suppose by symmetry that this subsequence is increasing. These nonmonotone intervals are then separated from each other by at least one point in part B . Applying the Erdős-Szekeres Theorem to those points we can find a monotone subsequence of length at least m . As each nonmonotone interval in part A contains 21 , we have therefore concluded that every $(3, 1)$ -alternation of length at least $4m^4$ contains a (symmetry of a) subpermutation of length at least $3m$ which can be divided by a single vertical line into parts A and B in which:

- part A consists of an increasing set of intervals order isomorphic to 21 , each separated from every other by at least one point in part B and
- part B consists of a monotone set of points, each separated from every other by at least one interval in A .

The following proposition then follows immediately.

Proposition A.6. *If the permutation class \mathcal{C} contains arbitrarily long $(3, 1)$ -alternations then it contains a symmetry of one of either*

$$\text{Grid}(\oplus 21 \text{ Av}(21)) \text{ or } \text{Grid}(\oplus 21 \text{ Av}(12)).$$

As with the previous cases, it suffices to compute the lower growth rates of these two classes, which are equal to each other and to the lower growth rates of the gridded permutations in them by Proposition 2.1. Thus we count only the gridded permutations in $\text{Grid}(\oplus 21 \text{ Av}(21))$.

We encode these permutations using the alphabet $\{\ell_1, \ell_{21}, r\}$ where ℓ_1 denotes a single entry in the left-hand section, ℓ_{21} denotes an interval order isomorphic to 21 in this section, and r denotes a single entry in the right-hand section, reading from bottom to top. For example, the permutation shown in Figure 11 corresponds to the word $\ell_{21}rr\ell_1\ell_{21}r\ell_{21}rrr\ell_1\ell_1$. In this manner, the gridded permutations in this class correspond precisely to the language $\{\ell_1, \ell_{21}, r\}^*$. To enumerate these permutations we assign a weight of x to the letters ℓ_1 and r and a weight of x^2 to ℓ_{21} and compute the weight-generating function for this language to be $1/(1 - 2x - x^2)$, giving the following.

Proposition A.7. *If the permutation class \mathcal{C} contains arbitrarily long $(3, 1)$ -alternations then $\text{gr}(\mathcal{C}) \geq 1 + \sqrt{2} \approx 2.41421$.*

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